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Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation II: Partial Differential Equations (PDEs)

Exercise Sheet 2- Difference Approximation, Order of Consistency

Return of Exercise Sheet: May 3 (before the lecture); Lab-Exercise until May 10

Homework 4: *θ–Scheme* (4 Points)

Show that the θ–Scheme from the Lecture course for $\theta = 0.5$ and

$$\varphi_j^n = f(x_j, t_n + k/2)$$

yields an approximation of order $O(h^2 + k^2)$ for solutions $u \in C^{4,3}$ of the heat equation. Which order of approximation is obtained when using this scheme with

$$\varphi_j^n = f(x_j, t_n) \ ?$$

What does this mean practically?

Homework 5: *Difference Approximation and Order of Consistency* (3 Points)

Construct a difference approximation for the differential operator

$$Lu := (\kappa(x)u')'$$

by using twice the approximation

$$u'(x) \approx \frac{1}{h}\left(u\Big(x+\frac{h}{2}\Big) - u\Big(x-\frac{h}{2}\Big)\right)$$

and study in detail the order of consistency.

Homework 6: *Consistency Error* (3 Points)

For the discretization of

$$-u''(x) = f(x), \quad u(0) = u(1) = 0$$

we choose an arbitrary grid $0 = x_0 < x_1 < \cdots < x_J$ and set the step size $h_j := x_j - x_{j-1}$.

1. Show that the following discretization is useful:

$$\frac{2}{h_j + h_{j+1}}\left(\frac{u_j - u_{j+1}}{h_{j+1}} + \frac{u_j - u_{j-1}}{h_j}\right) = f_j.$$

2. Compare the new consistency error to the one obtained on a uniform grid?

Lab-Exercise 1: *Temperature in a Slab*

The temperature distribution in a slab of length 1 satisfies the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0.$$

The temperature in both endpoints of the slab is given by

$$u(0,t) = u(1,t) = 12\sin^2(12\pi t), \quad t \geq 0.$$

Initially the slab has the temperature zero: $u(x,0) = 0, \quad 0 < x < 1.$

Compute the approximate temperature distribution with the *explicit Euler scheme* and with the *Crank–Nicolson scheme.*

1. Generate a plot using $h = 1/6$, $k = 1/72$ that shows the two computed solutions at time $t = 1/3$.
2. Plot the solutions using $h = 1/10$, $k = 1/72$ after the 1., 3. and 5. time step.
   Why does the solution of the explicit Euler scheme Verfahren stay initially zero at the interior grid points while the Crank–Nicolson solution is non-zero after the first time step?

Hint:
On `http://www-amna.math.uni-wuppertal.de/~ehrhardt/NumPar/matlab/ueb2.m` you will find a reference solution for $h = 1/10$, $k = 1/72$.

Alternatively you can determine a comparison solution of the heat equation with the web page `http://numawww.mathematik.tu-darmstadt.de/numerik_en/pdgl_en/heatflowtext.html`