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Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation II: Partial Differential Equations (PDEs)

Exercise Sheet 5- Poisson equation, maximum principle

Return of Exercise Sheet: June 7, 2012 (before the lecture)

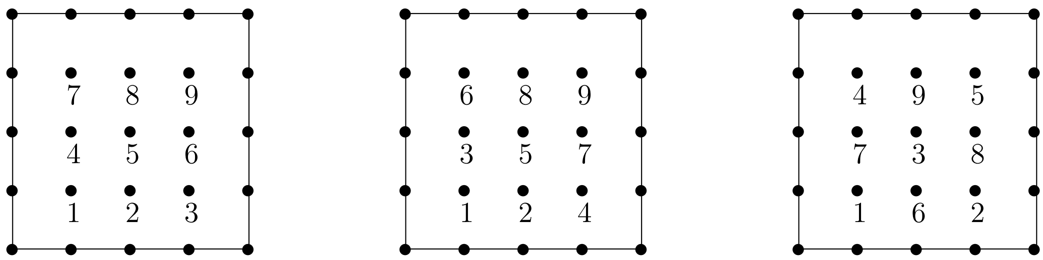
Homework 13: *Five Point Stencil* (4 Points)

The *Poisson equation* on the domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned} -\Delta u(x,y) &= f(x,y), \qquad && (x,y) \in \Omega \\ u(x,y) &= 0, && (x,y) \in \partial\Omega \end{aligned}$$

is discretized with the five point stencil and the step size $h = 1/4$.

1. State the system of equations for the following numerations:



i) row-wise ii) diagonal iii) chess board

2. Check for each case if the resulting system matrix A is symmetric.
3. Show that in case i) the matrix A is positive definite.

Homework 14: (3 Points)

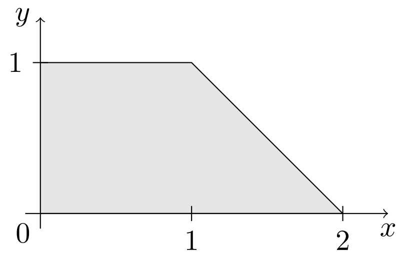
Prove the Corollaries 3.2 and 3.3 from Chapter 3.4 of the lecture course.

Homework 15: *Poisson Equation* (3 Points)

Consider the following boundary value problem

$$\begin{aligned} -\Delta u &= 1 \qquad && \text{in } \Omega \\ u &= 0 && \text{auf } \partial\Omega, \end{aligned}$$

where the domain Ω has the form



Derive, using the *maximum minimum principle* stated below, lower and upper bounds $m, R \in \mathbb{R}$ for the solution u:

$$m \le u(x,y) \le M \quad \forall (x,y) \in \Omega$$

Hint: Use the comparison function $v(x,y) = \dfrac{(2-x)x}{4} + \dfrac{(1-y)y}{4}$.

Definition: The linear differential operator

$$Lu := -\sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2}{\partial x_i \partial x_j}\,u + \sum_{i,j=1}^{n} b_i(x)\,\frac{\partial}{\partial x_i}\,u + cu, \quad x \in \Omega \subset \mathbb{R}^n$$

is *elliptic*, if the matrix $A(x) = (a_{ij}(x))_{ij}$ is symmetric and positive definite for all $x \in \Omega$ and if all coefficients are bounded. The operator L is *uniformly elliptic*, if there exists a constant C_E, such that

$$-\xi^\top A(x)\xi \ge C_E \|\xi\|^2, \qquad \text{for all} \quad x \in \Omega,\ \xi \in \mathbb{R}^n.$$

Maximum Minimum Principle: Let Ω be a bounded domain and L a uniformly elliptic operator with $c \ge 0$. Further assume that the data is continuous: $f \in C(\bar{\Omega})$, $\mu \in C(\partial\Omega)$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ are classical solutions to

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega \\ u &= \mu \qquad \text{on } \partial\Omega, \end{aligned}$$

Then

1. $f \ge 0 \quad \Longrightarrow \quad u \ge \min_{x \in \partial\Omega} \mu$,
2. $f \le 0 \quad \Longrightarrow \quad u \le \max_{x \in \partial\Omega} \mu$,
3. $f = 0 \quad \Longrightarrow \quad |u| \le \max_{x \in \partial\Omega} |\mu|$,
4. f arbitrary $\quad \Longrightarrow \quad |u| \le \max_{x \in \partial\Omega} |\mu| + K \max_{x \in \Omega} |f|$.

Lab-Exercise 2: *Poisson Equation*

Discretize the *Poisson equation*

$$\begin{aligned} -\Delta u(x,y) &= 3x^2, && (x,y) \in \Omega = (-1,1)^2 \\ u(x,y) &= 0, && (x,y) \in \partial\Omega, \quad x \ne 1 \\ u(1,y) &= \frac{1}{4}\sin\Big(\frac{y+1}{2}\,\pi\Big), && y \in (-1,1) \end{aligned}$$

using the five point stencil with the step size $h = \Delta x = \Delta y = 1/8$. Compute the appproximate solution and plot it.

Let $N = 1/h$. Compute the appproximate solutions for $N = 4, 8, 16, 32, ...$ until the computing time gets too high. Determine the ℓ^∞-error between the appproximate solution on the finest grid and the solutions on the coarser grids (for simplicity only in the grid points of the coarser grid), i.e.

$$e_N = \|u_{N_{\max}} - u_N\|_{\ell^\infty(\Omega_N)} \qquad N = 4, 8, ...$$

and create a log-log plot of this error vs. N. Add to this plot a line for N^{-2}.

Hint:
A comparison solution can determined by solving the Helmholtz equation (with $\lambda = 0$) using the webpage `http://numawww.mathematik.tu-darmstadt.de/numerik_en/pdgl_en/helmholtztext.html`.