Absorbing boundary conditions for nonlinear Schrödinger equations

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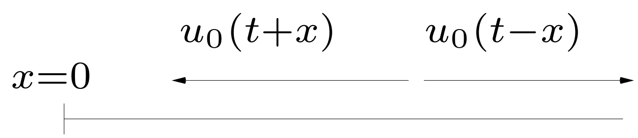
ABC for linear variable coefficients equations (Engquist-Majda)

1D wave equation : $\partial_t^2 - \partial_x^2 = -(\partial_x + \partial_t)(\partial_x - \partial_t)$.

We annihilate the reflected wave : $(\partial_x - \partial_t)u = 0$.



$$\partial_t^2 - \partial_x^2 + \alpha(t,x) + \beta(t,x)\partial_t + \gamma(t,x)\partial_x$$
$$= -(\partial_x - a(x,t,D_t))(\partial_x - b(x,t,D_t)) + S^{-\infty}$$

where $a(x,t,D_t)$ and $b(x,t,D_t)$ are pseudodifferential operators

Transparent boundary condition : $(\partial_x - b(x,t,D_t))u = 0$

ABC for linear variable coefficients equations (Engquist-Majda)

$a(x,t,\tau) = \sum_{j\geq 0} a_{1-j}(x,t,\tau)$ and $b(x,t,\tau) = \sum_{j\geq 0} b_{1-j}(x,t,\tau)$

$$\begin{cases} b_1 = i\tau \text{ and } a_1 = -i\tau, \\ b_0 = \dfrac{\gamma-\beta}{2} \text{ and } a_0 = \dfrac{\beta+\gamma}{2}, \\ b_{-1} = \dfrac{-\alpha + (\beta^2-\gamma^2)/4 + (\partial_x - \partial_t)(\gamma-\beta)/2}{2i\tau} = -a_{-1}. \end{cases}$$

The k-th order absorbing boundary condition is

$$\left(\partial_x - \sum_{j=0}^{k} b_{1-j}(0,t,D_t)\right) u|_{x=0} = 0.$$

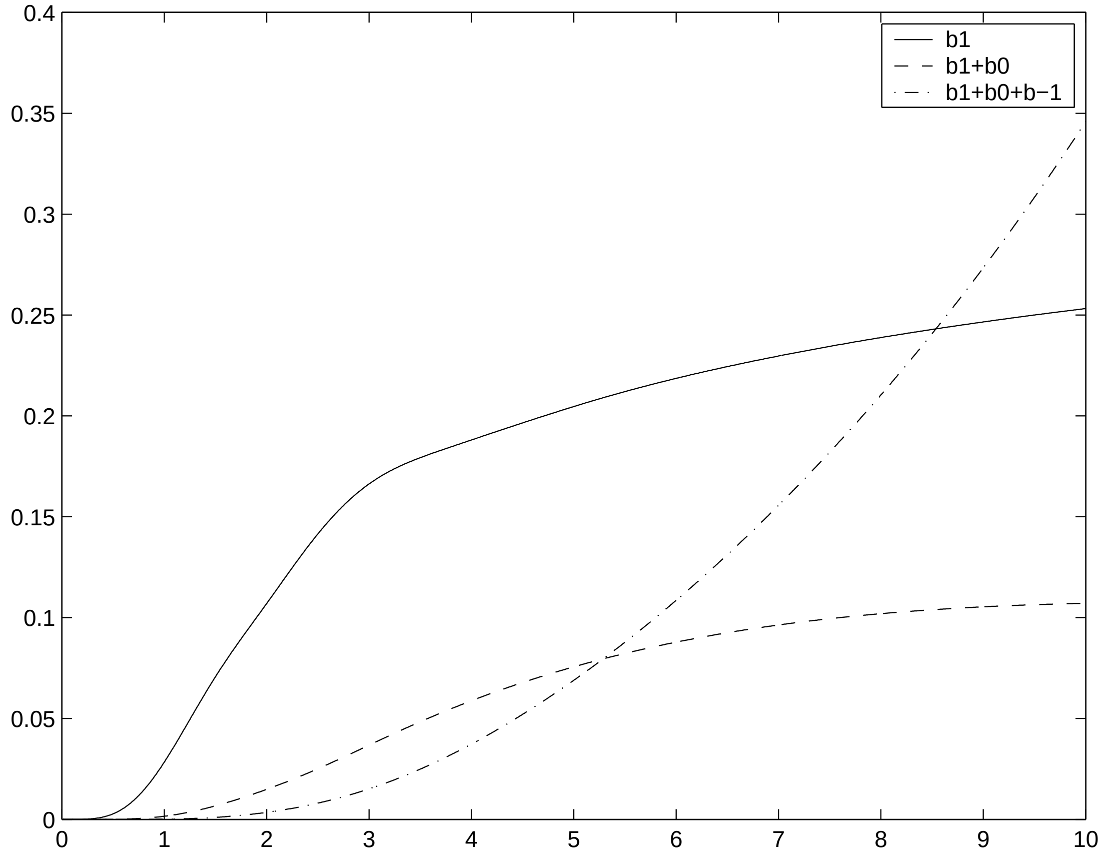


FIG. 1 – $\partial_t^2 u - \partial_x^2 u + \partial_t u = 0$. Relative error in L^2 in function of time. abc — order 0, $--$ order 1 and $\cdot - \cdot$ order 2

Short-time behaviour of ABC

High frequency asymptotic expansions

Numerical frequencies $= [\pi/T, \pi/\Delta t]$

$T \ll 1 \Rightarrow$ only high frequencies

Thus, ABC always good for small times

Goal : We would like to recover the properties of the Engquist-Majda method in the nonlinear case

Potential strategy

Linearization strategy

Numerical results

Potential strategy

$$(i\partial_t + \partial_x^2)u + |u|^2 u = 0$$

$$V(t,x) = |u(t,x)|^2$$

$$(i\partial_t + \partial_x^2)u + Vu = 0$$

We can see this as a linear equation with a potential term and apply the Engquist-Majda method

$\Rightarrow$ We obtain BC involving V

ABC of order 0 : $\partial_x u + \sqrt{-i\partial_t} u = 0$

Potential strategy

$$(i\partial_t + \partial_x^2)u + |u|^2 u = 0$$

ABC order 1 : $\partial_x u + \sqrt{-i\partial_t}\, u = 0$

ABC order 2 : $\partial_x u + \sqrt{-i\partial_t}\, u + |u|^2/2\sqrt{-i\partial_t}^{-1} u = 0$

ABC order 3 :
$$\partial_x u + \sqrt{-i\partial_t}\, u + |u|^2/2\sqrt{-i\partial_t}^{-1} u + i\partial_x(|u|^2)/4\partial_t^{-1} u = 0$$

$\sqrt{-i\partial_t}$ and $\sqrt{-i\partial_t}^{-1}$ are approximated using quadrature formulas

Other models : reaction-diffusion, semi-linear wave equation

Linearization strategy

$$(i\partial_t + \partial_x^2)u + u\partial_x u = 0$$

Linearize around u : $(i\partial_t + \partial_x^2)v + u\partial_x v + \partial_x u v = 0$

This is a linear equation for v with coefficients $u(t,x)$ and $\partial_x u(t,x)$

We may apply the Engquist-Majda method

$\Rightarrow$ We obtain BC for v involving u and $\partial_x u$

We have then to 'unlinearize'

For example, $(\partial_x + \sqrt{-i\partial_t})v + uv/2 = 0$

is unlinearized to give the first order ABC :

$$(\partial_x + \sqrt{-i\partial_t})u + u^2/4 = 0$$

Linearization strategy

$$(i\partial_t + \partial_x^2)u + u\partial_x u = 0$$

ABC order 0 : $\partial_x u + \sqrt{-i\partial_t}u = 0$

ABC order 1 : $\partial_x u + \sqrt{-i\partial_t}u + u^2/4 = 0$

ABC order 2 : $\partial_x u + \sqrt{-i\partial_t}u + u^2/8 - \sqrt{-i\partial_t}^{-1}(u\partial_x u/4) = 0$

Problem of the linearization strategy :

It is not always clear how to 'unlinearize'

Other model : semi-linear wave equation

Numerical results

$(i\partial_t + \partial_x^2)u + |u|^2 u = 0$: relative error

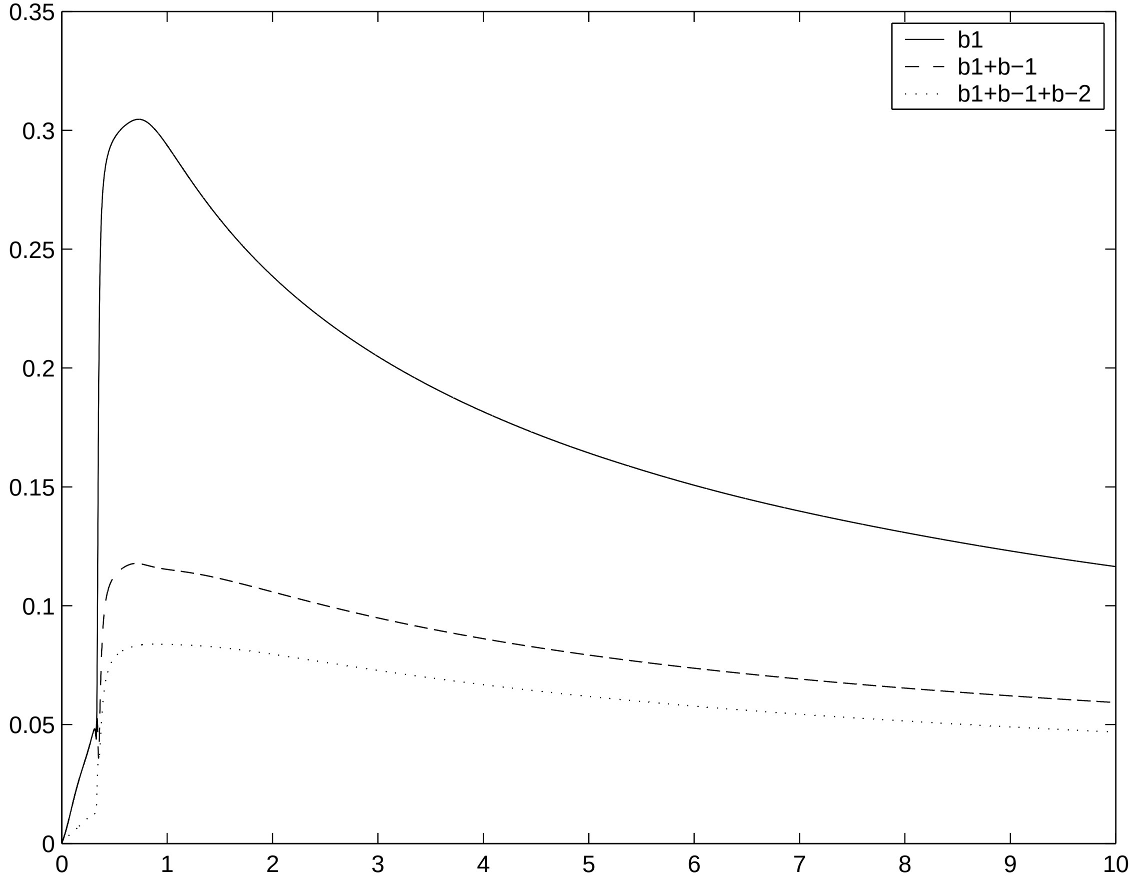


FIG. 2 – Potential strategy —— order 0, – – order 2 and ··· order 3.

$(i\partial_t + \partial_x^2)u + u\partial_x u = 0$: relative error

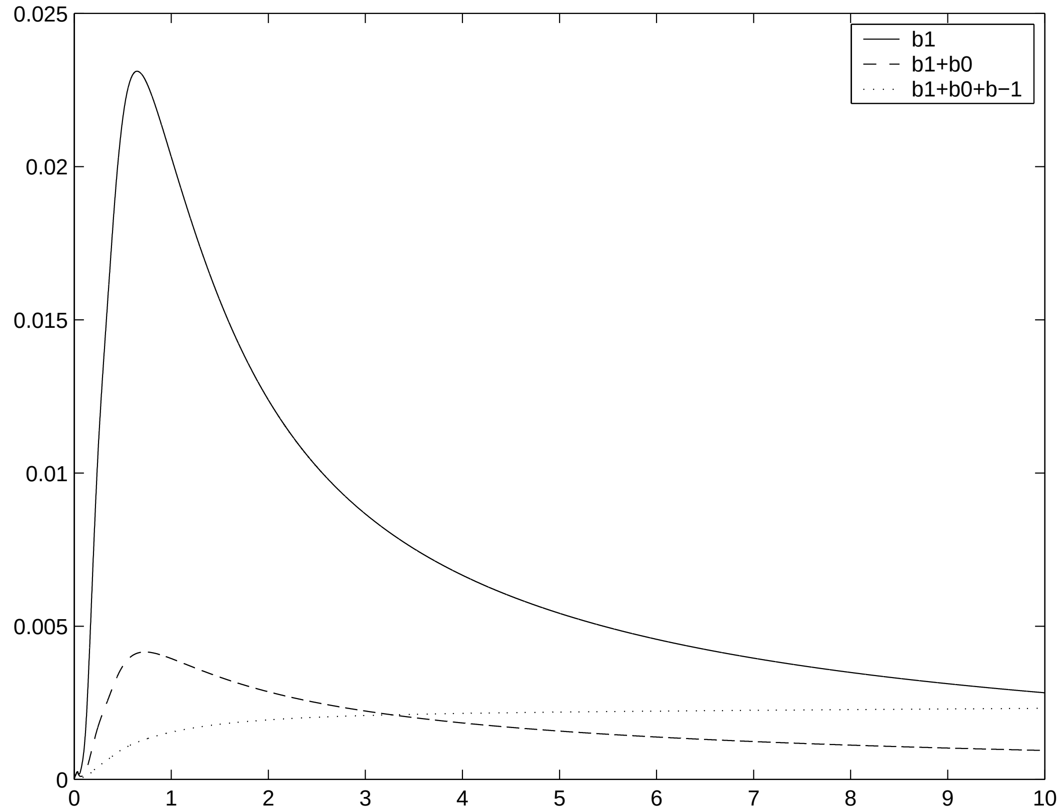


FIG. 3 – Linearization strategy —— order 0, – – order 1 and $\cdots$ order 2.

$(i\partial_t + \partial_x^2)u + u\partial_x u = 0$: **relative error**

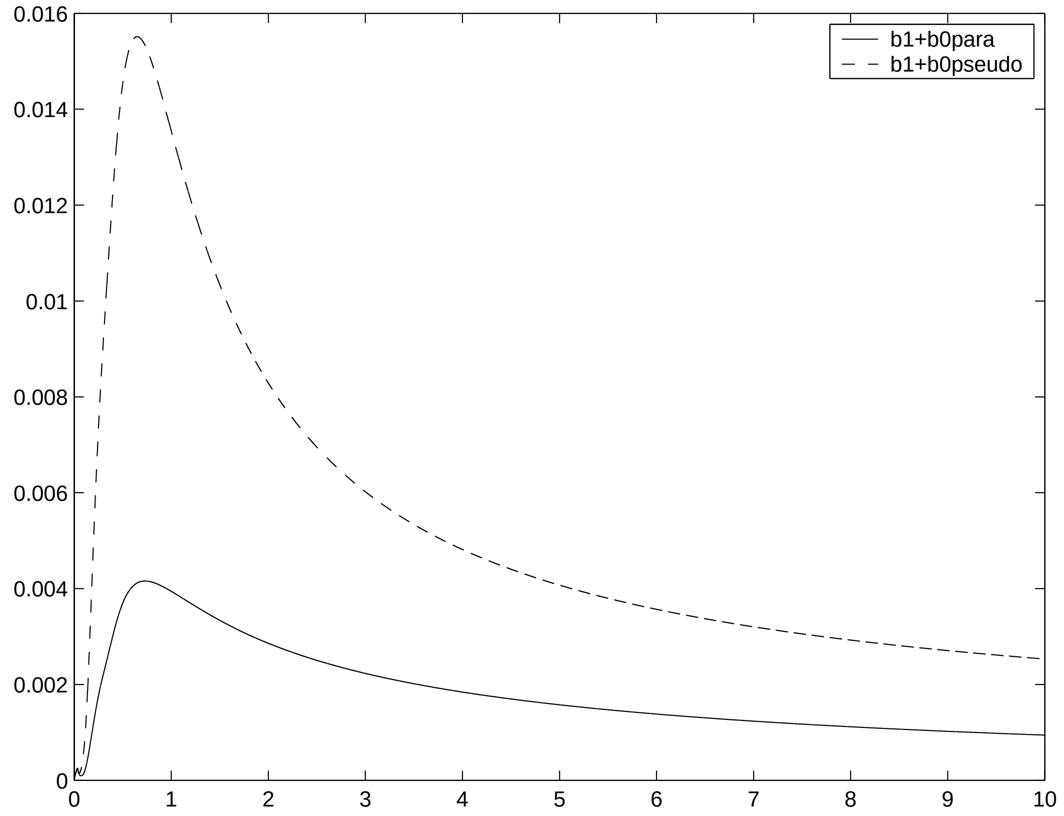


FIG. 4 – Linearization strategy —— order 1, and potential strategy – – order 1.

$(i\partial_t + \partial_x^2)u + u\partial_x u = 0$: relative error

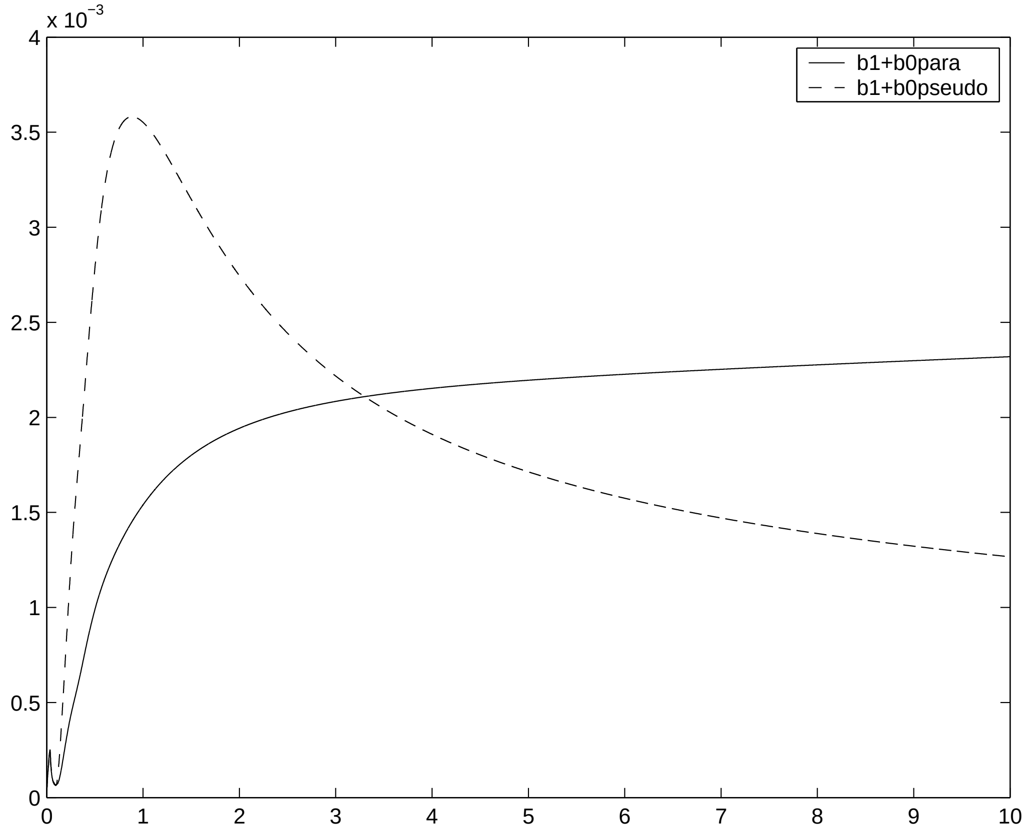


FIG. 5 – Linearization strategy —— order 2, and potential strategy – – order 2.

Optimality of the ABC

$$\begin{cases} (i\partial_t + \partial_x^2)u + u\partial_x u = 0 \text{ in }]0,T[\times]0,2[\\ \partial_x u - \sqrt{-i\partial_t}u - \alpha\, u^2 = 0 \text{ at } x = 0 \\ \partial_x u + \sqrt{-i\partial_t}u + \alpha\, u^2 = 0 \text{ at } x = 2 \end{cases}$$

order zero : $\alpha = 0$, order 1 with potential strategy : $\alpha = 1/2$, and order 1 with linearization strategy : $\alpha = 1/4$

$\alpha = 0$	$\alpha = \frac{1}{8}$	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\alpha = -\frac{1}{4}$
0.0231	0.0136	0.0042	0.0155	0.0551	0.1390	0.3463	0.0601

TAB. 1 – Maximum of the relative error for $0 \leq t \leq 10$ and for various choices of α.

Conclusions and perspectives

- We have extended the method of B. Engquist and A. Majda to nonlinear problems in two ways : the potential strategy and the linearization strategy

- The linearization strategy behaves better

- Improve the method for long-time computations

- Extend the results to dimension d and curved boundaries