Nonlinear Models in Option Pricing – an Introduction

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Abstract. Nonlinear Black–Scholes equations have been increasingly attracting interest over the last two decades, since they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, risks from an unprotected portfolio, large investor's preferences or illiquid markets, which may have an impact on the stock price, the volatility, the drift and the option price itself.

This book consists of a collection of contributed chapters of well–known outstanding scientists working successfully in this challenging research area. It discusses concisely several models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a volatility depending on different factors, such as the stock price, the time, the option price and its derivatives. We will present in this book both analytical techniques and numerical methods to solve adequately the arising nonlinear equations.

The purpose of this book is to give an overview on the current state-of-the-art research on nonlinear option pricing. The intended audience is on the one hand graduate and Ph.D. students of (mathematical) finance and on the other hand lecturer of mathematical finance and and people working in banks and stock markets that are interested in new tools for option pricing.

1 Introduction

Nonlinear models im mathematical finance are becoming more and more important since they take into account effects like the presence of transaction costs, feedback and illiquid market effects due to large traders choosing given stock-trading strategies, imperfect replication and investor's preferences and risk from unprotected portfolios.

Due to transaction costs, illiquid markets, large investors or risks from an unprotected portfolio the assumptions in the classical Black–Scholes model become unrealistic and the model results in strongly or fully nonlinear, possibly degenerate, parabolic diffusion–convection equations, where the stock price, volatility, trend and option price may depend on the time, the stock price or the option price itself.

In this chapter we will be concerned with several models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a volatility depending on different factors, such as the stock price, the time, the option price and its derivatives, where the nonlinearity results from the presence of transaction costs.

In the following sections we will give a short introduction to option pricing.

2 Financial Derivatives

The interest in pricing financial derivatives – among them in pricing options – arises from the fact that financial derivatives, also called contingent claims, can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called *hedging*. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this introductory chapter we will focus on European and American Call and Put options.

Definition 2.1 *A* European Call option *is a contract where at a prescribed time in the future, known as the expiry or expiration date T ($t = 0$ means 'today'), the holder of the option may purchase a prescribed asset, known as the underlying asset or the underlying $S(t)$, for a prescribed amount, known as the exercise or strike price K. The opposite party, or the writer, has the obligation to sell the asset if the holder chooses to buy it.*

At the final time T the holder of the European Call option will check the current price of the underlying asset $S := S(T)$. If the price of the asset is greater than the strike price, $S \geq K$, then the holder will exercise the Call and buy the stock for the strike price K. Afterwards, the holder will immediately sell the asset for the price S and make a profit of $V = S - K$. In this case the *cash flow*, or the difference of the money received and spent, is positive and the option is said to be *in-the-money*. If $S = K$, the cash flow resulting from an immediate exercise of the option is zero and the option is said to be *at-the-money*. In case $S \leq K$, the cash flow is negative and the option is said to be *out-of-the-money*. In the last two cases the holder will not exercise the Call option, since the asset S can be purchased on the market for K or less than K, which makes the Call option worthless. Therefore, the value of the European Call option at expiry, known as the *pay-off function*, is

$$V(S,T) = (S - K)^+,$$

with the notation $f^+ = \max(f, 0)$.

Definition 2.2 *Reciprocally, a* European Put option *is the right to sell the underlying asset $S(t)$ at the expiry date T for the strike price K. The holder of the Put may exercise this option, the writer has the obligation to buy it in case the holder chooses to sell it.*

The Put is in-the-money if $K \geq S$, at-the-money if $K = S$ and out-of-the-money if $K \leq S$. The pay-off function for a European Put option is therefore

$$V(S,T) = (K - S)^+.$$

The pay-off functions for the European Call and Put option are plotted in Fig, 1 from the perspective of the holder. This perspective is called the *long position*. The perspective of the writer, or the *short position*, is reversed and can be seen when the pay-off functions in Fig. 1 are multiplied by -1. That means that the writer of a European Call option is taking the risk of a potentially unlimited loss and must carefully design a strategy to compensate for this risk [27].

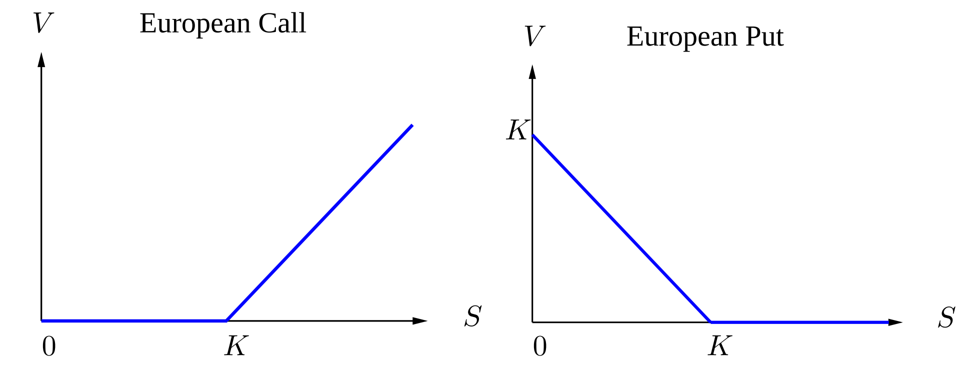


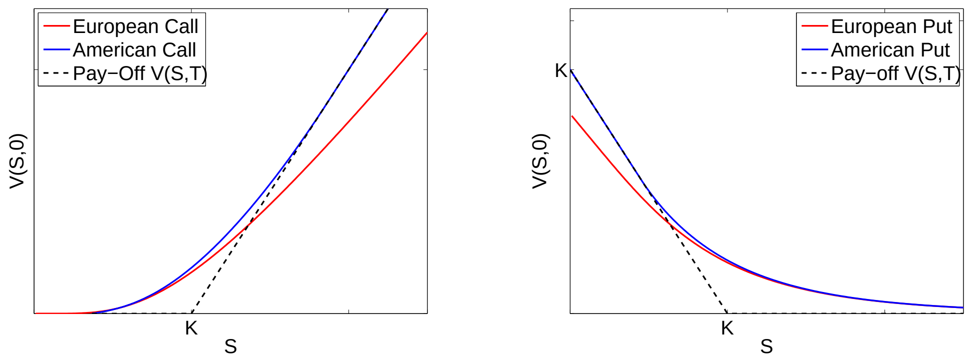
Figure 1: Pay-off functions for European options with a strike price K.

While European options can only be exercised at the expiry date T, *American options* can be exercised at any time until the expiration. Since an American option includes at least the same rights as the corresponding European option, the value of an American option V^{am} can never be smaller than the value of a European option V^{eur}, i.e.

$$V^{\mathrm{am}} \geq V^{\mathrm{eur}}.$$

Whether the values are equal depends on the *dividend yield* q, which describes the percentage rate of the returns on the underlying asset. Assuming that the underlying stock S pays no dividends, the values of a European and an American Call option are equal if all the other parameters remain the same (for details see [13, 34]). In case of an American Put option without dividend payments it can often be advantageous to exercise it before expiry, so that the values of a European and an American Put can differ substantially.

In the presence of a continuous dividend payment the fair price $V(S,0)$ of both an American Call and Put option is greater than the value of a European Call or Put, see Fig. 2.

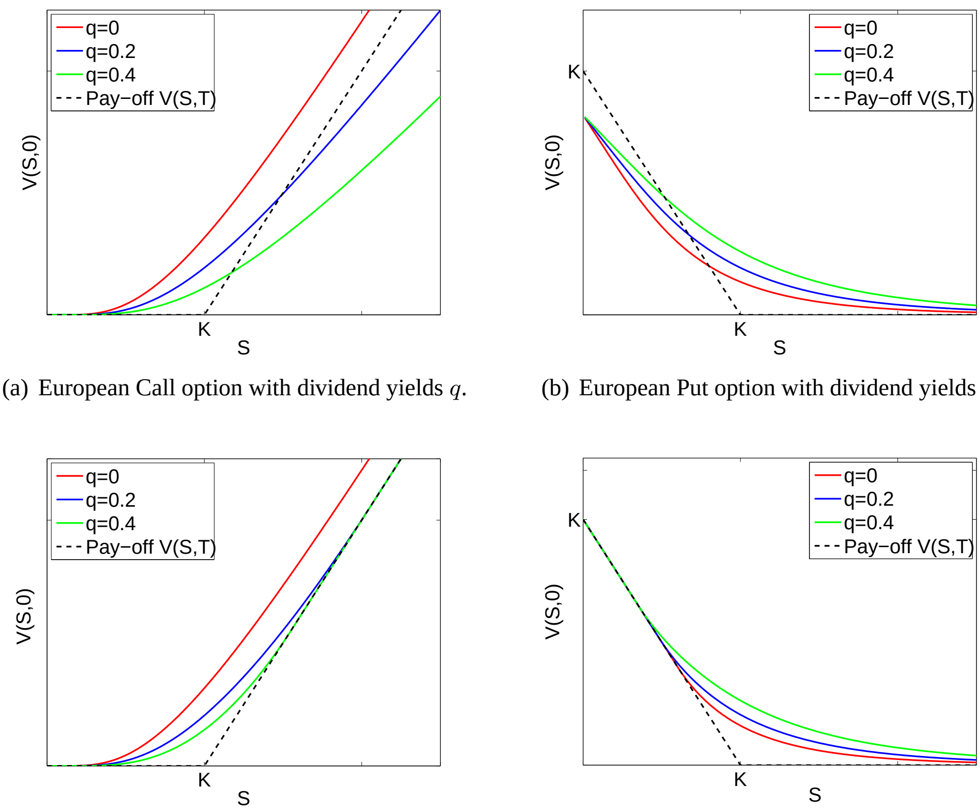


(a) American vs. European Call with dividends. (b) American vs. European Put option.

Figure 2: Schematical values of American vs. European options at $t = 0$.

Furthermore, it should be mentioned that the value of a Call option on an underlying without a dividend payment is always greater than the value of a Call option on an underlying with a dividend payment for both European and American options. For European and American Put options on an underlying without a dividend payment the value is less than on an underlying with a dividend payment. The influence of a dividend payment is summarized in Fig. 3.

Options, whose pay-offs only depend on the final value of the underlying asset, are called *vanilla* options. Options, whose pay-offs depend on the path of the underlying asset, are called *exotic* or *path-dependent* options. Examples are *Asian*, *Barrier* and *lookback* options. In this chapter, we will be solely concerned with plain vanilla European and American options.



(a) European Call option with dividend yields q.

(b) European Put option with dividend yields q.

(c) American Call option with dividend yields q.

(d) American Put option with dividend yields q.

Figure 3: The influence of a dividend yield.

3 Linear Black–Scholes Equations

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [3] in 1973 and previously by Robert Merton in [25]. The solution of the famous *(linear) Black–Scholes equation* [9]

$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \tag{1}$$

where $S := S(t) > 0$ and $t \in (0, T)$, provides both an option pricing formula for a European option and a hedging portfolio that replicates the contingent claim assuming that [27]:

- The price of the asset price or underlying asset S follows a Geometric Brownian motion, meaning that if $W := W(t)$ is a standard Brownian motion (see Appendix A.6), then S satisfies the following stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dW.$$

- The *trend* or *drift* μ (measures the average rate of growth of the asset price), the *volatility* σ (measures the standard deviation of the returns) and the riskless interest rate r are constant for $0 \le t \le T$ and no dividends are paid in that time period.

- The market is *frictionless*, thus there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may take any real number. Moreover, individual trading will not influence the price.

- There are no *arbitrage* opportunities, meaning that there are no opportunities of instantly making a risk-free profit ("There is no such thing as free lunch").

Under these assumptions the market is *complete*, which means that any derivative and any asset can be replicated or hedged with a portfolio of other assets in the market (see [31]). Then, it is well-known that the linear Black–Scholes equation (1) can be transformed into the heat equation and analytically solved to price the option [33]. The derivation of the solution can be found in [27], the formulae for the European Call and Put options are attached in Appendix B.

For American options, in general, analytic valuation formulae are not available, except for a few special types, which we are not going to address in this chapter. Those types are Calls on an asset that pays discrete dividends and *perpetual* Calls and Puts – meaning Calls and Puts with an infinite time to expiry [23].

4 Nonlinear Black–Scholes Equations

It is quite easy to imagine that the restrictive assumptions mentioned in the previous Section 3 are never fulfilled in reality. Due to transaction costs (cf. [2, 5, 24]), large investor preferences (cf. [11, 12, 26]) and incomplete markets (cf. [30]) these assumptions are likely to become unrealistic and the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic convection–diffusion equations, where both the volatility σ and the drift μ can depend on the time t, the stock price S or the derivatives of the option price V itself.

In this chapter we will focus on several transaction cost models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a constant drift μ and a nonconstant *modified volatility function*

$$\widetilde{\sigma}^2 := \widetilde{\sigma}^2(t, S, V_S, V_{SS}).$$

Under these circumstances (1) becomes the following *nonlinear Black–Scholes equation*, which we will consider for European options:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV, \tag{2}$$

where $dS = \mu S dt + \widetilde{\sigma} S dW$, $S > 0$ and $t \in (0, T)$.

Studying the linear Black–Scholes equation (1) for an American Call option would be redundant, since the value of an American Call option equals the value of a European Call option if no dividends are paid and the volatility is constant.

In order to make the model more realistic, we will consider a modification of the nonlinear Black–Scholes equation (2) for American options, where S pays out a *continuous dividend* $qSdt$ in a time step dt:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r - q)SV_S - rV, \tag{3}$$

where S follows the dynamics $dS = (\mu - q)S dt + \widetilde{\sigma} S dW$, $S > 0$, $t \in (0, T)$ and the dividend yield q is constant.

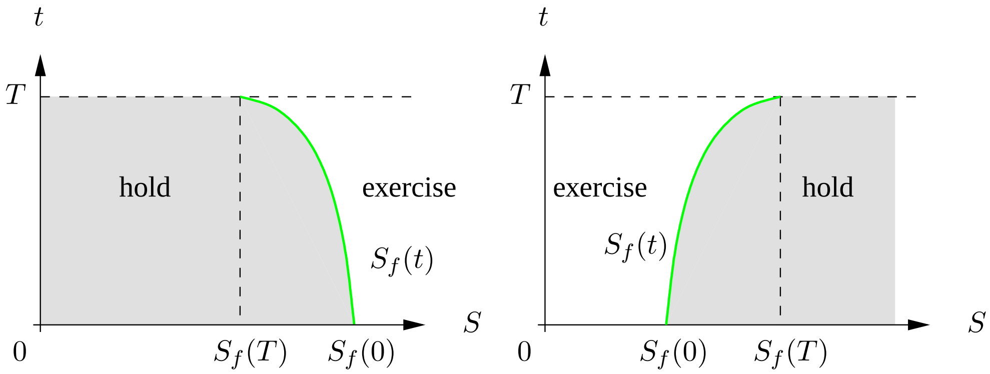
In the mathematical sense the nonlinear Black–Scholes equations (2) and (3) are called convection–diffusion equations. The second-order term $\frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS}$ is responsible for the *diffusion*, the first-order term rSV_S or $(r - q)SV_S$ is called the *convection* term and $-rV$ can be interpreted as the *reaction* term (see [27, 32]).

In the financial sense, the partial derivatives indicate the sensitivity of the option price V to the corresponding parameter and are called *Greeks*. The option delta is denoted by $\Delta = V_S$, the option gamma by $\Gamma = V_{SS}$ and the option theta by $\theta = V_t$. For a detailed discussion of this issue we refer to [19].

5 Terminal and Boundary Conditions

In order to find a unique solution for the equation (2) we need to complete the problem by stating the terminal and boundary conditions for both the European Call and Put option.

Since American options can be exercised at any time before expiry, we need to find the optimal time t of exercise, known as the *optimal exercise time*. At this time, which mathematically is a *stopping time* (see Appendix A.5), the asset price reaches the *optimal exercise price* or *optimal exercise boundary* $S_f(t)$. This leads to the formulation of the problem for American options by dividing the domain $[0,\infty[\times[0,T]$ of (3) into two parts along the curve $S_f(t)$ and analyzing each of them (see Fig. 4). Since $S_f(t)$ is not known in advance but has to be determined in the process of the solution, the problem is called *free boundary value problem* [34].



(a) American Call. (b) American Put.

Figure 4: Exercising and holding regions for American options.

For different numerical approaches, the free boundary problem for American options can be reformulated into a *linear complementary problem* (LCP), a *variational inequality* and a *minimization problem* [13]. The most simple treatment is the formulation as a free boundary problem [8, 14].

Even though we will focus on Call options in this chapter, we state the conditions for Put options for the sake of completeness.

5.1 European Call Option

The value $V(S,t)$ of the European Call option is the solution to (2) on $0 \le S < \infty$, $0 \le t \le T$ with the following terminal and boundary conditions:

$$\begin{aligned} V(S,T) &= (S-K)^+ && \text{for } 0 \le S < \infty \\ V(0,t) &= 0 && \text{for } 0 \le t \le T \\ V(S,t) &\sim S - Ke^{-r(T-t)} && \text{as } S \to \infty. \end{aligned} \tag{4}$$

5.2 European Put Option

Reciprocally, the value $V(S,t)$ of the European Put option is the solution to (2) on $0 \le S < \infty$, $0 \le t \le T$ with the pay–off function for the Put as the terminal condition and the boundary conditions:

$$
\begin{aligned}
V(S,T) &= (K-S)^+ && \text{for } 0 \le S < \infty \\
V(0,t) &= Ke^{-r(T-t)} && \text{for } 0 \le t \le T \\
V(S,t) &\to 0 && \text{as } S \to \infty.
\end{aligned} \tag{5}
$$

5.3 American Call Option

For the American Call option the *spatial* domain is divided into two regions by the *free boundary* $S_f(t)$, the *stopping region* $S_f(t) < S < \infty$, $0 \le t \le T$, where the option is exercised or dead with $V(S,t) = S - K$ and the *continuation region* $0 \le S \le S_f(t)$, $0 \le t \le T$, where the option is held or stays alive and (3) is valid under the following terminal and boundary conditions (see Fig. 4(a)):

$$
\begin{aligned}
V(S,T) &= (S-K)^+ && \text{for } 0 \le S \le S_f(T) \\
V(0,t) &= 0 && \text{for } 0 \le t \le T \\
V(S_f(t),t) &= S_f(t) - K && \text{for } 0 \le t \le T \\
V_S(S_f(t),t) &= 1 && \text{for } 0 \le t \le T \\
S_f(T) &= \max(K, rK/q).
\end{aligned} \tag{6}
$$

For the sake of simplicity we will assume $r > q$ in this chapter, and therefore we have $S_f(T) = rK/q$ for the American Call.

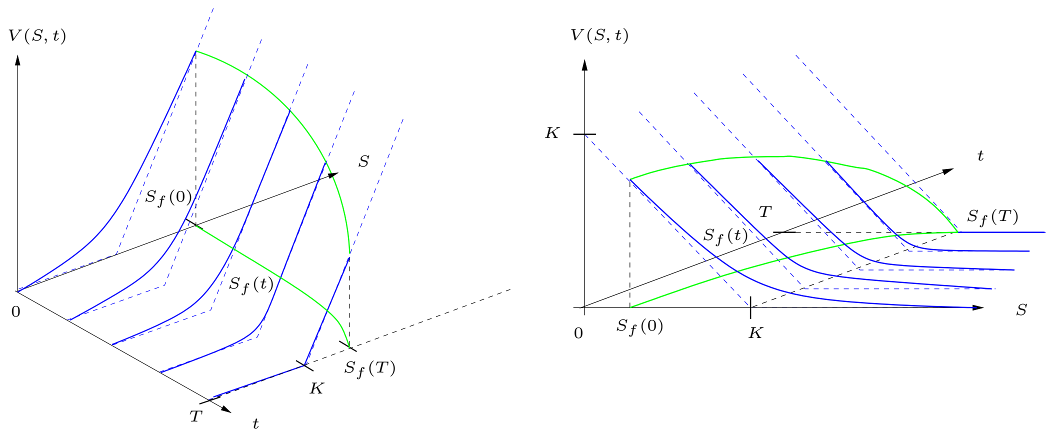
The structure of the value of an American Call can be seen Fig. 5(a), where we notice that the free boundary $S_f(t)$ determines the position of the exercise. The exercising and holding regions are illustrated in Fig. 4(a).

5.4 American Put Option

The American Put option is exercised in the stopping region $0 \le S < S_f(t)$, $0 \le t \le T$ where it has the value $V(S,t) = K - S$ (see Fig. 4(b)). In the continuation region $S_f(t) \le S < \infty$, $0 \le t \le T$ the Put option stays alive and (3) is valid under the following terminal and boundary conditions:

$$
\begin{aligned}
V(S,T) &= (K-S)^+ && \text{for } S_f(T) \le S < \infty \\
\lim_{S\to\infty} V(S,t) &= 0 && \text{for } 0 \le t \le T \\
V(S_f(t),t) &= K - S_f(t) && \text{for } 0 \le t \le T \\
V_S(S_f(t),t) &= -1 && \text{for } 0 \le t \le T \\
S_f(T) &= \min(K, rK/q).
\end{aligned} \tag{7}
$$

Since we assumed that $r > q$, we have $S_f(T) = K$ for the American Put. In Fig. 5(b) one can see how the free boundary $S_f(t)$ determines the structure of an American Put.



(a) American Call. (b) American Put.

Figure 5: Schematical values $V(S,t)$ of American options.

6 Volatility Models

The essential parameter of the standard Black–Scholes model, that is not directly observable and is assumed to be constant, is the volatility σ. There have been many approaches to improve the model by treating the volatility in different ways and using a *modified volatility function* $\widetilde{\sigma}(\cdot)$ to model the effects of transaction costs, illiquid markets and large traders, which is the reason for the nonlinearity of (2) and (3). We will first give a brief overview of several volatility models and then focus on the volatility models of transaction costs.

- The constant volatility σ in the standard Black–Scholes model can be replaced by the estimated volatility from the former values of the underlying. This volatility is known as the *historical volatility* [13].

- If the price of the option and the other parameters are known, which is e.g. the case for the European Call and Put options (see Appendix B), then the *implied volatility* can be calculated from those Black–Scholes formulae. The implied volatility is the value σ, for which (24) or (25) is true compared to the real market data. It can be calculated implicitly via the difference between the observed option price V (from the market data) and the Black–Scholes formulae (24) or (25), where all the parameters – except for the implied volatility σ – are taken from the market data (the stock price S, the time t, the expiration date T, the strike price K, the interest rate r the dividend rate q).

  Considering options with different strike prices K but otherwise identical parameters, we see that the implicit volatility changes depending on the strike price. If the implicit volatility for a certain strike price K is less than the implicit volatility for both the strike price greater and less than K, this effect is called *volatility smile* [22].

- Replacing the constant volatility with the observed implicit volatilities at each stock price and time leads to the term of the *local volatility* $\widetilde{\sigma} := \widetilde{\sigma}(S,t)$. Dupire [7] examined the dependencies and expressed the local volatility as a function of implicit volatilities.

- Hull and White [18] and Heston [15] developed a model, in which the volatility follows the dynamics of a stochastic process. This is known as the *stochastic volatility*.

- The assumption, that each security is available at any time and any size, or that individual trading will not influence the price, is not always true. Therefore, illiquid markets and large trader effects have been modeled by several authors. In [11] Frey and Stremme and later Frey and Patie [12] considered these effects on the price and come up with the result
$$\widetilde{\sigma} = \frac{\sigma}{1-\rho\lambda(S)SV_{SS}}, \tag{8}$$
where σ the historical volatility, ρ constant, $\lambda(S)$ strictly convex function, $\lambda(S) \geq 1$. The function $\lambda(S)$ depends on the pay-off function of the financial derivative. For the European Call option, Frey and Patie show that $\lambda(S)$ is a smooth, slightly increasing function for $S \geq K$. Bordag and Chmakova [4] assumed that $\lambda(S)$ is constant and solve the problem (2) with the modified volatility (8) explicitly using Lie-group theory (see also [6]).

As the main scope of this general overview chapter, we draw our attention in the sequel to a more detailed description of several transaction cost models.

6.1 Transaction Costs

The Black–Scholes model requires a continuous portfolio adjustment in order to hedge the position without any risk. In the presence of transaction costs it is likely that this adjustment easily becomes expensive, since an infinite number of transactions is needed [23]. Thus, the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over- or underpriced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices.

It has been shown that in a market with transaction costs there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option (see [2]). Soner, Shreve and Cvitanič [29] proved that the minimal hedging portfolio that dominates a European Call is the trivial one (hence holding one share of the stock that the Call is written on), so that efforts have been made to find an alternate relaxation of the hedging conditions to better replicate the pay-offs of derivative securities.

6.2 The model of Leland

Leland's idea [24] of relaxing the hedging conditions is to trade at discrete times, which promises to reduce the expenses of the portfolio adjustment. He assumes that the transaction cost $\kappa|\Delta|S/2$, where κ denotes the round trip transaction cost per unit dollar of the transaction and Δ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S, is proportional to the monetary value of the assets bought or sold. Now consider a replicating portfolio with Δ units of the underlying and the *bond* B (a certificate of debt issued by a government or a corporation guaranteeing payment B plus interest by a specified future date):

$$\Pi = \Delta S + B.$$

After a small change in time of the size δt the change in the portfolio becomes

$$\delta\Pi = \Delta\delta S + rB\delta t - \frac{\kappa}{2}|\delta\Delta|S, \tag{9}$$

where δS is the change in price S, so that the first term represents the change in value, the second term represents the bond growth in δt time and $\delta\Delta$ represents the change in the number of assets, so that the last term becomes the transaction cost due to portfolio change.

We apply Itô's lemma (see 23 in Appendix A.7) to the value of the option $V := V(S,t)$ and get

$$\delta V = V_S\delta S + (V_t + \frac{\sigma^2}{2}S^2V_{SS})\delta t. \tag{10}$$

Assuming that the option V is replicated by the portfolio Π, their values have to match at all times and there can be no risk-free profit. With this no-arbitrage argument we get

$$\delta\Pi = \delta V.$$

Matching the terms in (9) and (10) we get $\Delta = V_S$ and

$$rB\delta t - \frac{\kappa}{2}|\delta\Delta|S = (V_t + \frac{\sigma^2}{2}S^2V_{SS})\delta t. \tag{11}$$

Leland shows that

$$\frac{\kappa}{2}|\delta\Delta|S = \frac{\sigma^2}{2}\,\mathrm{Le}\,S^2|V_{SS}|\delta t, \tag{12}$$

where Le denotes the *Leland number*, which is given by

$$\mathrm{Le} = \sqrt{\frac{2}{\pi}}\left(\frac{\kappa}{\sigma\sqrt{\delta t}}\right), \tag{13}$$

with δt being the transaction frequency (interval between successive revisions of the portfolio) and κ the round trip transaction cost per unit dollar of the transaction. Plugging (12) and $B = \Pi - \Delta S = V - SV_S$ into the equation (11) becomes

$$rV - rSV_S - \frac{\sigma^2}{2}\,\mathrm{Le}\,S^2|V_{SS}| = V_t + \frac{\sigma^2}{2}S^2V_{SS}. \tag{14}$$

Therefore, Leland deduces that the option price is the solution of the nonlinear Black–Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2 S^2 V_{SS} + rSV_S - rV,$$

with the *modified volatility*

$$\widetilde{\sigma}^2 = \sigma^2 \Big(1 + \mathrm{Le}\,\mathrm{sign}(V_{SS})\Big), \tag{15}$$

where σ represents the historical volatility and Le the Leland number. It follows from the definition of the Leland number (13) that the more frequent the rebalancing (δt smaller), the higher the transaction cost and the greater the value of V.

It is known that $V_{SS} > 0$ for European Puts and Calls in the absence of transaction costs. Assuming the same behavior in the presence of transaction costs, equation (2) becomes linear with an adjusted constant volatility $\widetilde{\sigma}^2 = \sigma^2(1 + \mathrm{Le}) > \sigma^2$.

Leland's model has played a significant role in financial mathematics, even though it has been partly criticized by e.g. Kabanov and Safarian in [21], who prove that Leland's result has a hedging error. The restriction of his model is the convexity of the resulting option price V (hence $V_{SS} > 0$) and the possibility to only consider one option in the portfolio. Hoggard, Whalley and Wilmott studied equation (2) with the modified volatility (15) for several underlyings in [17]. An extension to this approach to general pay-offs is obtained by Avellaneda and Parás [1].

6.3 Barles and Soner

In [2] Barles and Soner derived a more complicated model by following the above utility function approach of Hodges and Neuberger [16]. Consider the process of bonds owned $X(s)$ and the process of shares owned $Y(s)$. Let the trading strategy $\big(L(s), M(s)\big)$ be a pair of nondecreasing processes with $L(t) = M(t) = 0$, which are interpreted as the cumulative transfers, measured in shares of stock. $L(s)$ is measured in shares from bond to stock and $M(s)$ is measured in shares from stock to bond. Let $\kappa \in (0, 1)$ be the proportional transaction cost. The processes $X(s)$ and $Y(s)$ start with the initial values x and y, $s \in [t, T]$ and evolve according to

$$X(s) = x - \int_t^s S(\tau)(1+\kappa)\, dL(\tau) + \int_t^s S(\tau)(1-\kappa)\, dM(\tau) \tag{16}$$

and

$$Y(s) = y + L(s) - M(s). \tag{17}$$

The first integral in (16) represents buying shares of stock at a price increased by the proportional transaction cost, the second integral represents selling stock at a reduced price of the transaction cost. In (17) we add the amount of the stocks bought and subtract the amount for the stocks sold to the initial amount of stocks owned.

According to the utility maximization approach of Hodges and Neuberger [16], the price of a European Call option can be obtained as the difference between the maximum utility of the terminal wealth when there is no option liability and when there is such a liability. Following this approach, Barles and Soner considered two optimization problems. Let the *exponential utility function* be

$$U(\xi) = 1 - e^{-\gamma\xi}, \quad \xi \in \mathbb{R},$$

where $\gamma > 0$ is the *risk aversion factor*. The first value function is the expected utility from the final wealth without any option liabilities taken over the transfer processes

$$V_1(x, y, S(t), t) := \sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + Y(T)S(T)\big)],$$

the second one is the expected utility from the final wealth assuming that we have sold N European Call options taken over the transfer processes

$$V_2(x, y, S(t), t) := \sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + Y(T)S(T) - N(S(T) - K)^+\big)].$$

Hodges and Neuberger postulate that the price of each option is equal to the maximal solution Λ of the algebraic equation

$$\begin{aligned} V_2(x + N\Lambda, y, S(t), t) = \sup_{L(\cdot),M(\cdot)} & E[U\big(X(T) + N\Lambda + Y(T)S(T) \\ & - N(S(T) - K)^+\big)] \\ = \sup_{L(\cdot),M(\cdot)} & E[U\big(X(T) + Y(T)S(T)\big)] \\ = V_1&(x, y, S(t), t), \end{aligned}$$

which means that the option price Λ equals the increment of the initial capital at time t that is needed to cope with the option liabilities arising at T. By a linearity argument selling N options with risk aversion factor of γ yields the same price as selling one option with risk aversion factor γN. This leads to performing an asymptotic analysis as $\gamma N \to \infty$. Hence, we consider

$$U(\xi) = 1 - e^{-\gamma N\xi}$$

and

$$\varepsilon = \frac{1}{\gamma N}.$$

Then, we have

$$U_\varepsilon(\xi) = 1 - e^{-\frac{\xi}{\varepsilon}}, \quad \xi \in \mathbb{R}.$$

Our optimization problems become

$$V_1(x, y, S(t), t) = 1 - \inf_{L(\cdot),M(\cdot)} E[e^{-\frac{1}{\varepsilon}(X(T)+Y(T)S(T))}]$$

and

$$V_2(x,y,S(t),t) = 1 - \inf_{L(\cdot),M(\cdot)} E[e^{-\frac{1}{\varepsilon}(X(T)+Y(T)S(T)-(S(T)-K)^+)}].$$

For analysis simplification Barles and Soner define $z_{1,2} : \mathbb{R} \times (0,\infty) \times (0,T) \to \mathbb{R}$ by

$$V_1(x,y,S(t),t) = 1 - e^{-\frac{1}{\varepsilon}\left(x+yS(t)-z_1(y,S(t),t)\right)}$$

and

$$V_2(x,y,S(t),t) = 1 - e^{-\frac{1}{\varepsilon}\left(x+yS(t)-z_2(y,S(t),t)\right)}.$$

Then

$$z_1(y,S(t),T) = 0 \quad \text{and} \quad z_2(y,S(t),T) = (S(T)-K)^+$$

and the option price

$$\Lambda(x,y,S(t),t;\frac{1}{\varepsilon},1) = z_2(y,S(t),t) - z_1(y,S(t),t).$$

By the theory of stochastic optimal control [10], Barles and Soner state that the value functions V_1 and V_2 are the unique solutions of the dynamic programming equation

$$\min\{-V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S, -V_y + S(1+\kappa)V_x, V_y - S(1-\kappa)V_x\} = 0,$$

which leads to a dynamic programming equation for z_1 and z_2, which are independent of the variable x.

Supposing that the proportional transaction cost κ is equal to $a\sqrt{\varepsilon}$ for some constant $a > 0$, they prove that as $\varepsilon \to 0$ and $\kappa \to 0$

$$z_1 \to 0 \quad \text{and} \quad z_2 \to V,$$

where V is the unique (viscosity) solution of the nonlinear Black–Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2 S^2 V_{SS}^2 + rSV_S - rV,$$

where

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + \Psi(e^{r(T-t)}a^2 S^2 V_{SS})\right). \tag{18}$$

Here σ denotes the historical volatility, $a = \kappa/\sqrt{\varepsilon}$ and $\Psi(x)$ is the solution to the following nonlinear ordinary differential equation (ODE)

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)}-x},\ x \neq 0, \tag{19a}$$

with the initial condition

$$\Psi(0) = 0. \tag{19b}$$

The analysis of this ODE (19) by Barles and Soner in [2] implies that

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \Psi(x) = -1. \tag{20}$$

The property (20) encourages to treat the function $\Psi(\cdot)$ as the identity for large arguments and therefore to simplify the calculations. In this case the volatility becomes

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2 S^2 V_{SS}). \tag{21}$$

The existence of a viscosity solution to (2) for European options with the volatility given by (18) is proved by Barles and Soner in [2] and their numerical results indicate an economically significant price difference between the standard Black–Scholes model and the nonlinear model with transaction costs.

6.4 Risk Adjusted Pricing Methodology

In this model, proposed by Kratka in [22] and improved by Jandačka and Ševčovič in [20], the optimal time-lag δt between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected with the Risk Adjusted Pricing Methodology (RAPM) and the *modified volatility* is now of the form

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + 3\left(\frac{C^2 M}{2\pi} S V_{SS}\right)^{\frac{1}{3}}\right), \tag{22}$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ the risk premium measure.

It is worth mentioning that these nonlinear transaction cost models that are described above are all consistent with the linear model if the additional parameters for transaction costs are equal to zero and vanish (Le, $\Psi(\cdot)$, M).

Conclusion

In this chapter we provided a profound overview over nonlinear Black–Scholes equations for European and American options.

We introduced the reader to the financial terminology and to Black–Scholes equations and presented several reasons for their nonlinearity and focused on the nonlinearity resulting from a modified volatility function due to transaction costs. Here we focused on several transaction cost models, including Leland' model, Barles' and Soner's model, the identity model and the Risk Adjusted Pricing Methodology.

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Appendix

A Stochastics

In this chapter, we used several terms and concepts of probability theory and stochastics. Thus, we recall some definitions (see e.g. [13, 27, 28] and the references therein).

A.1 Probability Space

Let Ω be a *sample space* representing all possible scenarios (e.g. all possible paths for the stock price over time). A subset of Ω is an *event* and $\omega \in \Omega$ a *sample point*.

Definition A.1 *Let Ω be a nonempty set and $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a* σ-algebra *(not related to the volatility σ), if*

i) $\Omega \in \mathcal{F}$,

ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$ and

iii) whenever a sequence of sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

In our financial scenario, $\mathcal{F}$ represents the space of events that are observable in the market and therefore, all the information available until the time t can be regarded as a σ-algebra $\mathcal{F}_t$. It is logical that $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t < s$, since the information that has been available t is still available at s.

Definition A.2 *Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of Ω. A* probability measure *P is a function that assigns a number in $[0,1]$ to every set $A \in \mathcal{F}$. The number is called the* probability of *A and is written $P(A)$. We require:*

- $P(\Omega) = 1$ *and*
- *whenever a sequence of disjoint sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, then*

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

The tripel $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

A.2 Random Variable

Definition A.3 *A real-valued function X on Ω is called a* random variable *if the sets*

$$\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\} = X^{-1}(]-\infty, x])$$

are measurable for all $x \in \mathbb{R}$. That is, $\{X \leq x\} \in \mathcal{F}$.

A.3 Stochastic Process

Definition A.4 *A (continuous)* stochastic process $X(t) = X(\cdot, t)$, $t \in [0, \infty[$, *is a family of random variables* $X : \Omega \times [0, \infty[\to \mathbb{R}$ *with* $t \mapsto X(\omega, t)$ *continuous for all* $\omega \in \Omega$.

A.4 Itô Process

Definition A.5 *An* Itô process *is a stochastic process of the form*

$$dX = a(X, t)dt + b(X, t)dW,$$

which is equivalent to

$$X(t) = X(0) + \int_0^t a(X, s)ds + \int_0^t b(X, s)dW,$$

where $X(0)$ *is nonrandom,* $W(t)$ *is a standard Wiener process,* $a(\cdot)$ *and* $b(\cdot)$ *are sufficiently regular functions and the integrals are Itô integrals.*

A.5 Stopping Time

Definition A.6 *A* stopping time t *is a random variable taking values in* $[0, \infty]$ *and satisfying*

$$\{t \leq s\} \in \mathcal{F}_s \quad \forall s \geq 0.$$

A.6 Brownian Motion

Definition A.7 *A* Brownian motion *or* Wiener process *is a time-continuous stochastic process* $W(t)$ *with the properties:*

- $W(0) = 0$.
- $W(t) \sim \mathcal{N}(0, t)$ *for all* $t \geq 0$. *That is, for each* t *the random variable* $W(t)$ *is normally distributed with mean* $\mathrm{E}[W(t)] = 0$ *and variance* $\mathrm{Var}[W(t)] = \mathrm{E}[W^2(t)] = t$.
- *All increments* $\Delta W(t) := W(t + \Delta t) - W(t)$ *on non-overlapping time intervals are independent. That is,* $W(t_2) - W(t_1)$ *and* $W(t_4) - W(t_3)$ *are independent for all* $0 \leq t_1 < t_2 \leq t_3 < t_4$.
- $W(t)$ *depends continuously on* t.

A.7 Itô's Lemma

Theorem A.8 *Consider a function* $V(S,t) : \mathbb{R} \times [0,\infty[\to \mathbb{R}$ *with* $V \in C^{2,1}(\mathbb{R} \times [0,\infty[)$ *and suppose that* $S(t)$ *follows the Itô process*

$$dS = a(S,t)dt + b(S,t)dW,$$

where $W(t)$ *is a standard Wiener process. Then* V *follows an Itô process with the same Wiener process* $W(t)$*:*

$$dV = (aV_S + \frac{1}{2}b^2 V_{SS} + V_t)dt + bV_S dW, \tag{23}$$

where $a := a(S,t)$ *and* $b := b(S,t)$*.*

If we consider a special case, where $a(S,t) = \mu S$ and $b(S,t) = \sigma S$, then $S(t)$ follows the Geometric Brownian motion, where $W(t)$ is a standard Wiener process, and we have

$$dS = \mu S dt + \sigma S dW.$$

Then, Itô's Lemma yields

$$\begin{aligned} dV &= (\mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + V_t)dt + \sigma S V_S dW \\ &= \big(\frac{1}{2}\sigma^2 S^2 V_{SS} + V_t\big)dt + V_S dS. \end{aligned}$$

B Pricing Formulae

Theorem B.1 *The solution to the linear Black–Scholes equation* (1) *with the terminal and boundary conditions* (4)*, or the value of the* European Call *option, is given by*

$$V(S,t) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{24}$$

where

$$d_1 := \frac{\ln\big(\frac{S}{K}\big) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 := \frac{\ln\big(\frac{S}{K}\big) + (r - q - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and $\mathcal{N}(x)$ *is the standard normal cumulative distribution function*

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy, \qquad x \in \mathbb{R}.$$

Respectively, the value of the European Put *option is the solution to the linear Black–Scholes equation* (1) *with the terminal and boundary conditions* (5) *and is given by*

$$V(S,t) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2). \tag{25}$$

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