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The numerical solution of nonlinear Black–Scholes equations

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Abstract

Nonlinear Black–Scholes equations have been increasingly attracting interest over the last two decades, since they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, illiquid markets, risks from an unprotected portfolio or large investor's preferences, which may have an impact on the stock price, the volatility, the drift and the option price itself.
In this work we will be concerned with several models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a volatility depending on different factors, such as the stock price, the time, the option price and its derivatives, where the nonlinearity results from the presence of transaction costs.
In the European case we will consider a European Call option and analytically approach the option price by transforming the problem into a forward convection-diffusion equation with a nonlinear term. In case of American options we will consider an American Call option and transform this free boundary problem into a fully nonlinear parabolic equation defined on a fixed domain following Ševčovič's idea [72].
Finally, we will present the numerical results of different discretization schemes for European and American options for various volatility models including Leland's model, Barles' and Soner's model and the Risk Adjusted Pricing Methodology.

Zusammenfassung

Nichtlineare Black-Scholes Gleichungen sind im Laufe der letzten Jahre immer mehr in den Vordergrund gerückt, da sie eine genauere Optionspreisbestimmung ermöglichen, indem sie realistischere Annahmen treffen und Transaktionskosten, illiquide Märkte, Risiken durch ein ungeschütztes Portfolio oder die Effekte durch große Händler berücksichtigen. Diese Zusatzannahmen können im Black-Scholes-Modell sowohl den Aktienpreis, als auch die Volatilität, den Trend und den Optionspreis beeinflussen und dadurch die Modelleigenschaften verändern.

In dieser Arbeit konzentrieren wir uns auf die relevanteste Klasse der nichtlinearen Black-Scholes-Gleichungen für europäische und amerikanische Optionen, die die Transaktionskosten berücksichtigt. Diese können durch eine modifizierte Volatilität modelliert werden, so dass die Volatilität sowohl vom Aktienpreis und der Zeit, als auch vom dem Optionspreis und dessen Ableitungen abhängt. Dadurch wird die partielle Differentialgleichung nichtlinear.

Wir stellen diverse Volatilitätsmodelle zum Einbeziehen der Transaktionskosten vor - unter anderem das Modell von Leland, das Modell von Barles und Soner und die risikoangepasste Bewertungsmethode - und wenden diese auf eine europäische und amerikanische Call Option an.

Auf analytischer Ebene transformieren wir das Problem für die europäische Call Option in eine vorwärts Konvektions-Diffusions-Gleichung mit einem nichtlinearen Term. Im Falle der amerikanischen Call Option untersuchen wir das freie Randwertproblem und transformieren es in eine nichtlineare parabolische Gleichung auf einem festen Ortsgebiet.

Da diese Probleme keine analytische Lösung besitzen, stellen wir mehrere Diskretisierungsverfahren zu ihrer Lösung vor und lösen sie numerisch. Dabei konzentrieren wir uns auf die Methode der Finiten Differenzen.

Schliesslich präsentieren wir die Ergebnisse einiger klassischer und moderner kompakter Diskretisierungsschemata für mehrere Volatilitätsmodelle und vergleichen diese.

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Notation

Option Variables

t	time
T	expiry or expiration time
S, $S(t)$	price of the underlying asset at time t
K	strike price
V, $V(S,t)$	value of an option
V^{eur}	value of a European option
V^{am}	value of an American option
σ	constant volatility
$\widetilde{\sigma}(\cdot)$	modified (nonconstant) volatility function
μ	drift rate
r	riskless interest rate in the bank
q	dividend rate

Transformed Option Variables

τ	transformed time variable
$\widetilde{T}$	transformed expiry or expiration time
x	transformed spatial variable
u, $u(x,\tau)$	transformed option value for European options
Π, $\Pi(x,\tau)$	synthetic portfolio for American options

Abbreviations

DJIA	Dow Jones Industrial Average
S&P 500	Standard and Poor's 500
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation
RAPM	Risk Adjusted Pricing Methodology
PSOR	Projected Successive Over Relaxation

Mathematical Symbols

$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of real numbers > 0
$\mathbb{N}$	set of integers > 0
$\in$	element in
$:=$	defined to be

Mathematical Symbols continued

$\mapsto$	maps to
$\top$	transposed
$C^k([a,b])$	k times continuously differentiable functions
f^+	$:= \max(f,0)$
$\text{sign}(x)$	$:= \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$
$\mathcal{O}(h^r)$	Landau-symbol

Grid Variables

h	step size in the spatial variable
k	step size in the time variable
i	index for the spatial step ($x_i = ih$)
n	index for the time step ($\tau_n = nk$)

Chapter 1

Introduction

1.1 Financial Derivatives

The interest in pricing financial derivatives - among them in pricing options - arises from the fact that financial derivatives, also called contingent claims, can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called *hedging*. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this work we will focus on European and American Call and Put options.

Definition: *A **European Call option** is a contract where at a prescribed time in the future, known as the expiry or expiration date T, the holder of the option may purchase a prescribed asset, known as the underlying asset or the underlying $S(t)$, for a prescribed amount, known as the exercise or strike price K. The opposite party, or the writer, has the obligation to sell the asset if the holder chooses to buy it.*

At the final time T the holder of the European Call option will check the current price of the underlying asset $S := S(T)$. If the price of the asset is greater than the strike price, $S \geq K$, then the holder will exercise the Call and buy the stock for the strike price K. Afterwards, the holder will immediately sell the asset for the price S and make a profit of $V = S - K$. In this case the *cash flow*, or the difference of the money received and spent, is positive and the option is said to be *in-the-money*. If $S = K$, the cash flow resulting from an immediate exercise of the option is zero and the option is said to be *at-the-money*. In case $S \leq K$, the cash flow is negative and the option is said to be *out-of-the-money*. In the last two cases the holder will not exercise the Call option, since the asset S can be purchased on the market for K or less than K, which makes the Call option worthless. Therefore, the value of the European Call option at expiry, known as the *pay-off function*, is

$$V(S,T) = (S - K)^+.$$

Definition: *Reciprocally, a **European Put option** is the right to sell the underlying asset $S(t)$ at the expiry date T for the strike price K. The* *holder of the Put may exercise this option, the writer has the obligation to buy it in case the holder chooses to sell it (see e.g. [74]).*

The Put is in-the-money if $K \geq S$, at-the-money if $K = S$ and out-of-the-money if $K \leq S$. The pay-off function for a European Put option is therefore

$$V(S,T) = (K - S)^+.$$

The pay-off functions for the European Call and Put option are plotted in Figure 1.1 from the perspective of the holder. This perspective is called the *long position*. The perspective of the writer, or the *short position*, is reversed and can be seen when the pay-off functions in Figure 1.1 are multiplied by -1. That means that the writer of a European Call option is taking the risk of a potentially unlimited loss and must carefully design a strategy to compensate for this risk (see e.g. [61]).

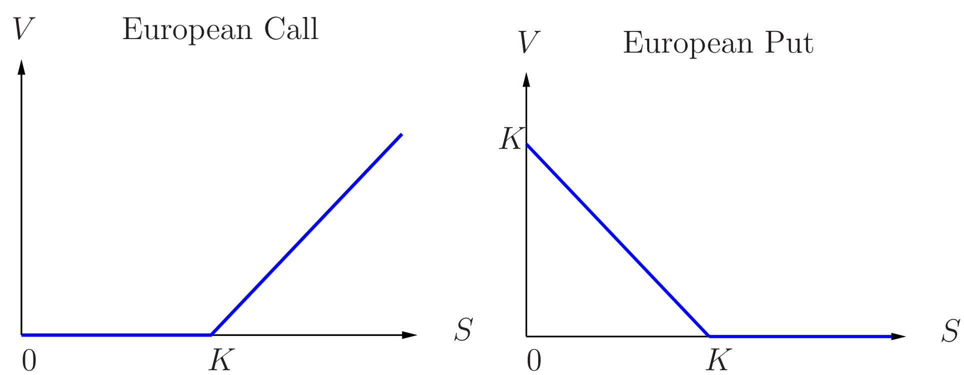


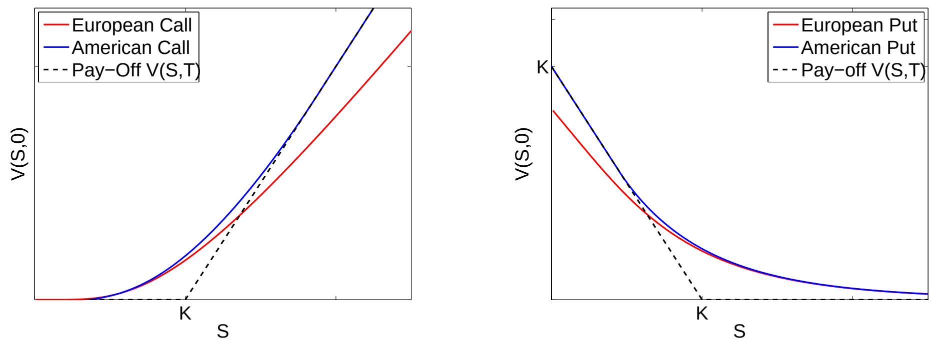
Figure 1.1: Pay-off functions for European options with a strike price K.

While European options can only be exercised at the expiry date T, *American options* can be exercised at any time until the expiration, which complicates their pricing process significantly.
It is a known consequence that the value of an American option V^{am} can never be smaller than the value of a European option V^{eur}, because an American option includes at least the same rights as the corresponding European option. That is:

$$V^{am} \geq V^{eur}.$$

Whether the values are equal depends on the *dividend yield* q, which describes the percentage rate of the returns on the underlying asset. Assuming that the underlying stock S pays no dividends, the values of a European and an American Call option are equal if all the other parameters remain the same (for details see [31, 78]). In case of an American Put option without dividend payments it can often be advantageous to exercise it before expiry, so that the values of a European and an American Put can differ substantially.

In the presence of a continuous dividend payment the fair price $V(S,0)$ of both an American Call and Put option is greater than the value of a European Call or Put. These facts are illustrated in Figure 1.2.



(a) American vs. European Call option in the presence of a dividend payment.

(b) American vs. European Put option.

Figure 1.2: Schematical values of American vs. European options at $t = 0$.

Furthermore, it should be mentioned that the value of a Call option on an underlying without a dividend payment is always greater than the value of a Call option on an underlying with a dividend payment for both European and American options. For European and American Put options on an underlying without a dividend payment the value is less than on an underlying with a dividend payment. The influence of a dividend payment is summarized in Figure 1.3.

Options, whose pay-offs only depend on the final value of the underlying asset, are called *vanilla* options. Options, whose pay-offs depend on the path of the underlying asset, are called *exotic* or *path-dependent* options. Examples are *Asian*, *Barrier* and *lookback* options. In this thesis, we will be concerned with plain vanilla European and American options.

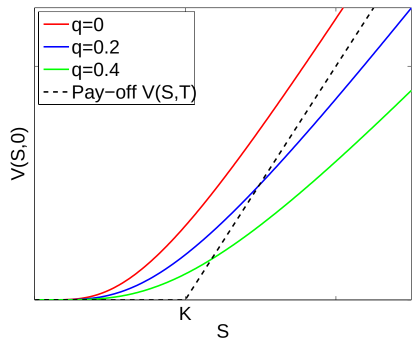
1.2 Linear Black-Scholes Equations

Option pricing theory has made a great leap forward since the development of the Black-Scholes option pricing model by Fischer Black and Myron Scholes in [7] in 1973 and previously by Robert Merton in [51]. The solution of the famous *(linear) Black-Scholes equation*

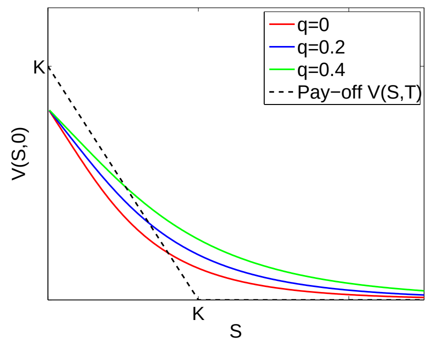
$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \qquad (1.1)$$

where $S := S(t) > 0$ and $t \in (0,T)$, provides both an option pricing formula for a European option and a hedging portfolio that replicates the contingent claim assuming that (see [61]):

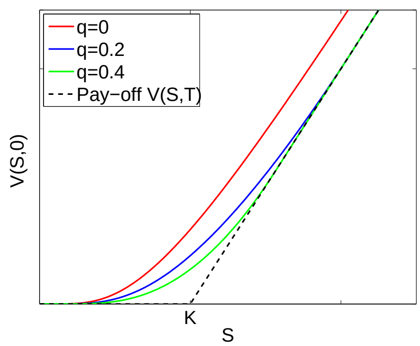
- The price of the asset price or underlying asset S follows a Geometric Brownian motion, meaning that if $W := W(t)$ is a standard Brownian



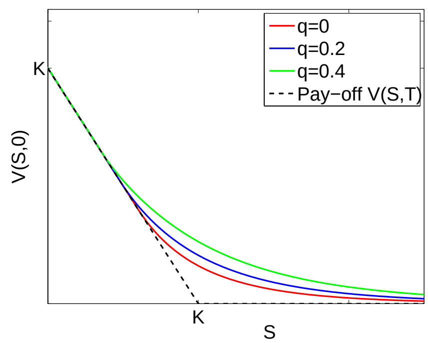
(a) European Call option with various dividend yields q.



(b) European Put option with various dividend yields q.



(c) American Call option with various dividend yields q.



(d) American Put option with various dividend yields q.

Figure 1.3: The influence of a dividend yield.

motion (see Appendix B.6), then S satisfies the following stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dW.$$

- The *trend* or *drift* μ (measures the average rate of growth of the asset price), the *volatility* σ (measures the standard deviation of the returns) and the riskless interest rate r are constant for $0 \leq t \leq T$ and no dividends are paid in that time period.

- The market is *frictionless*, thus there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may take any real number. Moreover, individual trading will not influence the price.

- There are no *arbitrage* opportunities, meaning that there are no opportunities of instantly making a risk-free profit ("There is no such thing as free lunch").

Under these assumptions the market is *complete*, which means that any derivative and any asset can be replicated or hedged with a portfolio of other assets in the market (see [69]). Then, it is well-known that the linear Black–Scholes equation (1.1) can be transformed into the heat equation and analytically solved to price the option [74]. The derivation of the solution can be found in [61], the formulae for the European Call and Put options are attached in Appendix C.
For American options, in general, analytic valuation formulae are not available, except for a few special types, which we are not going to address in this thesis. Those types are Calls on an asset that pays discrete dividends and *perpetual* Calls and Puts - meaning Calls and Puts with an infinite time to expiry [47]. For the other types, there are various kinds of analytical and numerical approximations that will be discussed in Chapter 4.

1.3 Nonlinear Black-Scholes Equations

It is easy to imagine that the restrictive assumptions mentioned in the previous section are never fulfilled in reality. Due to transaction costs (see [4, 9, 48]), large investor preferences (see [28, 29, 60]) and incomplete markets [64] they are likely to become unrealistic and the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic convection-diffusion equations (see Appendix A), where both the volatility σ and the drift μ can depend on the time t, the stock price S or the derivatives of the option price V itself. In this work we will be concerned with several transaction cost models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a constant drift μ and a nonconstant *modified volatility function*

$$\widetilde{\sigma}^2 := \widetilde{\sigma}^2(t, S, V_S, V_{SS}).$$

Under these circumstances (1.1) becomes the following *nonlinear Black-Scholes equation*, which we will consider for European options:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV, \qquad (1.2)$$

where $dS = \mu S dt + \widetilde{\sigma} S dW$, $S > 0$ and $t \in (0, T)$.
Studying (1.1) for an American Call option would be redundant, since the value of an American Call option equals the value of a European Call option if no dividends are paid and the volatility is constant. In order to make the model more realistic, we will consider a modification of (1.2) for American options, where S pays out a *continuous dividend* $qSdt$ in a time step dt:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r - q)SV_S - rV, \qquad (1.3)$$

where S follows the dynamics $dS = (\mu - q)Sdt + \widetilde{\sigma} SdW$, $S > 0$, $t \in (0, T)$ and the dividend yield q is constant.

Remark 1.1. *Most dividend payments on an index - such as the Dow Jones Industrial Average (DJIA) or the Standard and Poor's 500 (S&P500) - are so frequent that they can be modeled as a continuous payment, which is the case in* (1.3)*. However, if companies only make two or four dividend payments per year, then one has to treat the dividend payments discretely and the question of how to incorporate* ***discrete dividend payments*** *into the Black–Scholes equation arises.*
Even though in this work we will focus on the case of continuous dividend payments, we briefly review the results for discrete dividend payments from [75] in the sequel.
We assume that there is only one dividend payment of the dividend yield q during the lifetime of the option at the ***dividend date*** *t_q. Neglecting other factors, such as taxes, the asset price S must decrease exactly by the amount of the dividend payment q at time t_q. Thus we have the* ***jump condition***

$$S(t_q^+) = (1-q)S(t_q^-),$$

where t_q^-, t_q^+ denote the moments just before and after the dividend date t_q. This leads to the following effect on the option price:

$$V(S, t_q^-) = V((1-q)S, t_q^+), \tag{1.4}$$

i.e. the value of the option at S and time t_q^- is the same as the value immediately after the dividend date t_q but at the asset value $(1-q)S$. In order to calculate the value of a Call option with one dividend payment we solve the Black–Scholes equation from expiry $t = T$ until $t = t_q^+$ and use the relation (1.4) *to compute the values at $t = t_q^-$. Finally, we continue to solve the Black–Scholes equation backwards starting at $t = t_q^-$ using these values as the initial data. The boundary conditions, that are discussed in the next section, do not need to be modified for this case.*

In the mathematical sense equations (1.2) and (1.3) are called convection-diffusion equations. The second-order term $\frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2V_{SS}$ is responsible for the *diffusion*, the first-order term rSV_S or $(r-q)SV_S$ is the *convection* term and $-rV$ can be interpreted as the *reaction* term (see [61, 73]).
In the financial sense, the partial derivatives indicate the sensitivity of the option price V to the corresponding parameter and are called *Greeks*. The option delta is denoted by $\Delta = V_S$, the option gamma by $\Gamma = V_{SS}$ and the option theta by $\theta = V_t$ [37].

1.4 Terminal and Boundary Conditions

In order to find a unique solution for the equation (1.2) we need to complete the problem by stating the terminal and boundary conditions for both the European Call and Put option.
Since American options can be exercised at any time before expiry, we

need to find the optimal time t of exercise, known as the *optimal exercise time*. At this time, which mathematically is a *stopping time* (see Appendix B.5), the asset price reaches the *optimal exercise price* or *optimal exercise boundary* $S_f(t)$. This leads to the formulation of the problem for American options by dividing the domain $[0,\infty[\times[0,T]$ of (1.3) into two parts along the curve $S_f(t)$ and analyzing each of them (see Figure 1.4). Since $S_f(t)$ is not known in advance but has to be determined in the process of the solution, the problem is called *free boundary value problem* (see e.g. [78]). For different numerical approaches, the free boundary problem for American options can be reformulated into a *linear complementary problem*, a *variational inequality* and a *minimization problem* [31]. Here, we will only consider the formulation as a free boundary problem.
Even though we will focus on Call options in this thesis, we state the conditions for Put options for the sake of completeness.

1.4.1 European Call Option

The value $V(S,t)$ of the European Call option is the solution to (1.2) on $0 \leq S < \infty$, $0 \leq t \leq T$ with the following terminal and boundary conditions:

$$\begin{aligned} V(S,T) &= (S-K)^+ && \text{for } 0 \leq S < \infty \\ V(0,t) &= 0 && \text{for } 0 \leq t \leq T \\ V(S,t) &\sim S - Ke^{-r(T-t)} && \text{as } S \to \infty. \end{aligned} \tag{1.5}$$

1.4.2 European Put Option

Reciprocally, the value $V(S,t)$ of the European Put option is the solution to (1.2) on $0 \leq S < \infty$, $0 \leq t \leq T$ with the pay-off function for the Put as the terminal condition and the boundary conditions:

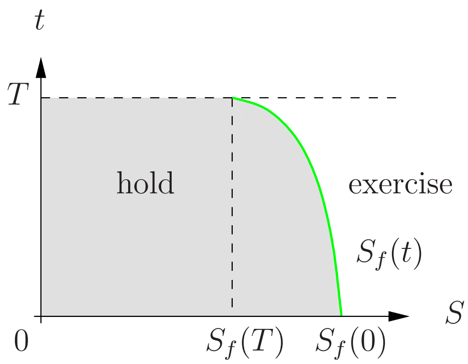
$$\begin{aligned} V(S,T) &= (K-S)^+ && \text{for } 0 \leq S < \infty \\ V(0,t) &= Ke^{-r(T-t)} && \text{for } 0 \leq t \leq T \\ V(S,t) &\to 0 && \text{as } S \to \infty. \end{aligned} \tag{1.6}$$

1.4.3 American Call Option

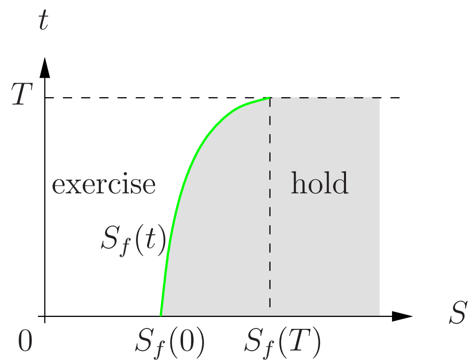
For the American Call option the *spatial* domain is divided into two regions by the *free boundary* $S_f(t)$, the *stopping region* $S_f(t) < S < \infty$, $0 \leq t \leq T$, where the option is exercised or dead with $V(S,t) = S - K$ and the *continuation region* $0 \leq S \leq S_f(t)$, $0 \leq t \leq T$, where the option is held or stays alive and (1.3) is valid under the following terminal and boundary conditions (see Figure 1.4(a)):

$$\begin{aligned} V(S,T) &= (S-K)^+ && \text{for } 0 \le S \le S_f(T) \\ V(0,t) &= 0 && \text{for } 0 \le t \le T \\ V(S_f(t),t) &= S_f(t) - K && \text{for } 0 \le t \le T \\ V_S(S_f(t),t) &= 1 && \text{for } 0 \le t \le T \\ S_f(T) &= \max(K, rK/q). \end{aligned} \tag{1.7}$$

For the sake of simplicity we will assume $r > q$ in this work, and therefore we have $S_f(T) = rK/q$ for the American Call.
The structure of the value of an American Call can be seen Figure 1.5(a), where we notice that the free boundary $S_f(t)$ determines the position of the exercise. The exercising and holding regions are illustrated in Figure 1.4(a).



(a) American Call.



(b) American Put.

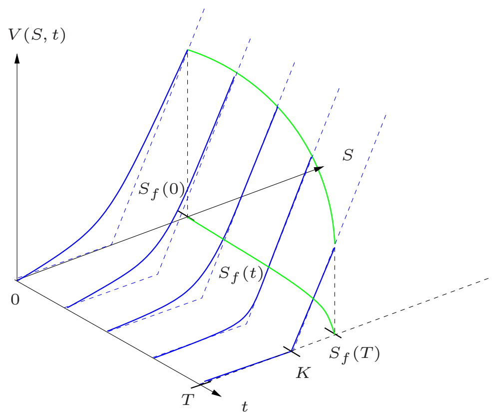
Figure 1.4: Exercising and holding regions for American options.

1.4.4 American Put Option

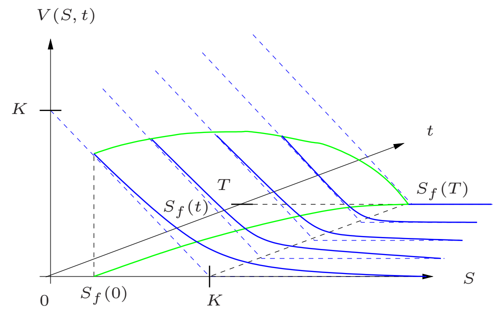
The American Put option is exercised in the stopping region $0 \le S < S_f(t)$, $0 \le t \le T$ where it has the value $V(S,t) = K - S$ (see Figure 1.4(b)). In the continuation region $S_f(t) \le S < \infty$, $0 \le t \le T$ the Put option stays alive and (1.3) is valid under the following terminal and boundary conditions:

$$\begin{aligned} V(S,T) &= (K-S)^+ && \text{for } S_f(T) \le S < \infty \\ \lim_{S\to\infty} V(S,t) &= 0 && \text{for } 0 \le t \le T \\ V(S_f(t),t) &= K - S_f(t) && \text{for } 0 \le t \le T \\ V_S(S_f(t),t) &= -1 && \text{for } 0 \le t \le T \\ S_f(T) &= \min(K, rK/q). \end{aligned} \tag{1.8}$$

Since we assumed that $r > q$, we have $S_f(T) = K$ for the American Put. In Figure 1.5(b) we can see how the free boundary $S_f(t)$ determines the structure of an American Put.



(a) American Call.



(b) American Put.

Figure 1.5: Schematical values $V(S,t)$ of American options.

1.5 Outline

The structure of this thesis is as follows. In the following chapter (Chapter 2) several nonconstant volatility models that lead to the nonlinearity of the Black–Scholes equation will be introduced. The focus of this work is the solution of the resulting nonlinear problems for European and American Call options. Since in general, a closed–form solution to the nonlinear Black–Scholes equation does not exist (for American options not even in the linear case), we have to solve the problems numerically.

This is achieved by initially analytically approaching the solution for the European Call by transforming (1.2) with (1.5) into a forward-in-time parabolic problem (see Section 3.1). In Section 4.1 thereafter both classical and compact finite-difference schemes will be specified and used to solve the transformed problem. Finally, different volatility models will be compared to each other.

The numerical solution and the comparison study for American options will be achieved by the transformation of the free boundary problem (1.3) subject to (1.7) into a parabolic equation defined on a fixed spatial domain (Section 3.2). This new problem will be numerically solved by the method of finite differences using an operator splitting technique. It will then be evaluated and concisely discussed in Section 4.2 thereafter.

Chapter 2

Volatility Models

The essential parameter of the standard Black–Scholes model, that is not directly observable and is assumed to be constant, is the volatility σ. There have been many approaches to improve the model by treating the volatility in different ways and using a *modified volatility function* $\widetilde{\sigma}(\cdot)$ to model the effects of transaction costs, illiquid markets and large traders, which is the reason for the nonlinearity of (1.2) and (1.3). In this section we will first give a brief overview of several volatility models and then focus on the volatility models of transaction costs.

- The constant volatility σ in the standard Black–Scholes model can be replaced by the estimated volatility from the former values of the underlying. This volatility is known as the *historical volatility* [31].

- If the price of the option and the other parameters are known, which is e.g. the case for the European Call and Put options (see Appendix C), then the *implied volatility* can be calculated from those Black–Scholes formulae. The implied volatility is the value σ, for which (C.1) or (C.2) is true compared to the real market data. It can be calculated implicitly via the difference between the observed option price V (from the market data) and the Black–Scholes formulae (C.1) or (C.2), where all the parameters - except for the implied volatility σ - are taken from the market data (the stock price S, the time t, the expiration date T, the strike price K, the interest rate r the dividend rate q).
  Considering options with different strike prices K but otherwise identical parameters, we see that the implicit volatility changes depending on the strike price. If the implicit volatility for a certain strike price K is less than the implicit volatility for both the strike price greater and less than K, this effect is called *volatility smile* (see e.g. [46]).

- Replacing the constant volatility with the observed implicit volatilities at each stock price and time leads to the term of the *local volatility* $\widetilde{\sigma} := \widetilde{\sigma}(S,t)$. Dupire [19] examines the dependencies and expresses the local volatility as a function of implicit volatilities.

- Hull and White [38] and Heston [33] develop a model, in which the volatility follows the dynamics of a stochastic process. This is known as the *stochastic volatility*.

- The assumption, that each security is available at any time and any size, or that individual trading will not influence the price, is not always true. Therefore, illiquid markets and large trader effects have been modeled by several authors. In [29] Frey and Stremme and later Frey and Patie [28] consider these effects on the price and come up with the result
$$\widetilde{\sigma} = \frac{\sigma}{1 - \rho\lambda(S)SV_{SS}}, \tag{2.1}$$
where σ the historical volatility, ρ constant, $\lambda(S)$ strictly convex function, $\lambda(S) \geq 1$. The function $\lambda(S)$ depends on the pay-off function of the financial derivative. For the European Call option, Frey and Patie show that $\lambda(S)$ is a smooth, slightly increasing function for $S \geq K$. Bordag and Chmakova [8] assume that $\lambda(S)$ is constant and solve the problem (1.2) with the modified volatility (2.1) explicitly using Lie-group theory (see also [12, 23]).

The main scope of this thesis is the numerical solution of nonlinear Black–Scholes equations resulting from transaction costs. Therefore, after this general overview, we devote our attention to a more detailed description of several transaction cost models.

2.1 Transaction Costs

The Black–Scholes model requires a continuous portfolio adjustment in order to hedge the position without any risk. In the presence of transaction costs it is likely that this adjustment easily becomes expensive, since an infinite number of transactions is needed [47]. Thus, the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over- or underpriced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices. It has been shown that in a market with transaction costs there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option (see [4]). Soner, Shreve and Cvitanič proved in [63] that the minimal hedging portfolio that dominates a European Call is the trivial one (hence holding one share of the stock that the Call is written on), so that efforts have been made to find an alternate relaxation of the hedging conditions to better replicate the pay-offs of derivative securities.

2.2 Leland

Leland's idea of relaxing the hedging conditions is to trade at discrete times [48], which promises to reduce the expenses of the portfolio adjustment. He assumes that the transaction cost $\kappa|\Delta|S/2$, where κ denotes the round trip transaction cost per unit dollar of the transaction and Δ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S, is proportional to the monetary value of the assets bought or sold. Now consider a replicating portfolio with Δ units of the underlying and the *bond* B (a certificate of debt issued by a government or a corporation guaranteeing payment B plus interest by a specified future date):

$$\Pi = \Delta S + B.$$

After a small change in time of the size δt the change in the portfolio becomes

$$\delta\Pi = \Delta\delta S + rB\delta t - \frac{\kappa}{2}|\delta\Delta|S, \tag{2.2}$$

where δS is the change in price S, so that the first term represents the change in value, the second term represents the bond growth in δt time and $\delta\Delta$ represents the change in the number of assets, so that the last term becomes the transaction cost due to portfolio change.
We apply Itô's lemma (see B.1 in Appendix B.7) to the value of the option $V := V(S,t)$ and get

$$\delta V = V_S\delta S + (V_t + \frac{\sigma^2}{2}S^2V_{SS})\delta t. \tag{2.3}$$

Assuming that the option V is replicated by the portfolio Π, their values have to match at all times and there can be no risk-free profit. With this no-arbitrage argument we get

$$\delta\Pi = \delta V.$$

Matching the terms in (2.2) and (2.3) we get $\Delta = V_S$ and

$$rB\delta t - \frac{\kappa}{2}|\delta\Delta|S = (V_t + \frac{\sigma^2}{2}S^2V_{SS})\delta t. \tag{2.4}$$

Leland shows that

$$\frac{\kappa}{2}|\delta\Delta|S = \frac{\sigma^2}{2}LeS^2|V_{SS}|\delta t, \tag{2.5}$$

where Le denotes the *Leland number*, which is given by

$$Le = \sqrt{\frac{2}{\pi}}\left(\frac{\kappa}{\sigma\sqrt{\delta t}}\right), \tag{2.6}$$

with δt being the transaction frequency (interval between successive revisions of the portfolio) and κ the round trip transaction cost per unit dollar of the transaction. Plugging (2.5) and $B = \Pi - \Delta S = V - SV_S$ into the equation (2.4) becomes

$$rV - rSV_S - \frac{\sigma^2}{2} LeS^2|V_{SS}| = V_t + \frac{\sigma^2}{2} S^2 V_{SS}. \tag{2.7}$$

Therefore, Leland deduces that the option price is the solution of the nonlinear Black-Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2 S^2 V_{SS} + rSV_S - rV,$$

with the *modified volatility*

$$\widetilde{\sigma}^2 = \sigma^2 \Big(1 + Le\, \mathrm{sign}(V_{SS})\Big), \tag{2.8}$$

where σ represents the historical volatility and Le the Leland number. It follows from the definition of the Leland number (2.6) that the more frequent the rebalancing (δt smaller), the higher the transaction cost and the greater the value of V.

It is known that $V_{SS} > 0$ for European Puts and Calls in the absence of transaction costs. Assuming the same behavior in the presence of transaction costs, equation (1.2) becomes linear with an adjusted constant volatility $\widetilde{\sigma}^2 = \sigma^2(1 + Le) > \sigma^2$.

Leland's model has played a significant role in financial mathematics, even though it has been partly criticized by e.g. Kabanov and Safarian in [44], who prove that Leland's result has a hedging error. The restriction of his model is the convexity of the resulting option price V (hence $V_{SS} > 0$) and the possibility to only consider one option in the portfolio. Hoggard, Whalley and Wilmott study equation (1.2) with the modified volatility (2.8) for several underlyings in [35]. An extension to this approach to general payoffs is obtained by Avellaneda and Parás in [3].

2.3 Parás and Avellaneda

From the binomial model making use of the algorithm of Bensaid et al. (see [6]), Parás and Avellaneda derive the same volatility (2.8) as Leland. Dropping the convexity assumption of the resulting option price they state that in case $V_{SS} \leq 0$ and $Le \geq 1$ (hence $\widetilde{\sigma} \leq 0$) the problem (1.2) becomes mathematically ill-posed and does not possess a solution for general pay-off functions [3]. For the case $V_{SS} > 0$ and $Le \geq 1$ (hence $\widetilde{\sigma} > 0$) they propose several hedging strategies.

2.4 Boyle and Vorst

Using the central limit theorem in [9] Boyle and Vorst derive from the binomial model with transaction costs and discrete trading processes that as the time step δt and the transaction cost κ tend to zero, the price of the option converges to a Black-Scholes price with the *modified volatility* of the form

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + Le\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(V_{SS})\right). \tag{2.9}$$

Just like Leland, Boyle and Vorst assume convexity of V, so that $\widetilde{\sigma}^2 = \sigma^2(1 + Le\sqrt{\pi/2})$ and (1.2) turns into a linear equation. However, here δt in the definition of Le (2.6) is the mean time length for a change in the value of the stock, not the transaction frequency (see [4]).

2.5 Hodges and Neuberger

In [34] Hodges and Neuberger suggest a different approach to model transaction costs. They consider a *utility function*, without specifying it, and assume that the behavior of the investor is characterized by this function. The utility function measures the relative satisfaction of the investor from the input. They show that the Black–Scholes price is the difference between the maximum utility from the final wealth with and without option liability. They postulate that the price of the option in a market with transaction costs should be equal to the unique cash increment which offsets this difference. This theory in the presence of transaction costs is further developed by Davis, Panas and Zariphopoulou in [17]. Constantinides and Zariphopoulou [13] modify this original definition of the price and obtain universal bounds independent of the utility function.

2.6 Barles and Soner

In [4] Barles and Soner derive a more complicated model by following the above utility function approach of Hodges and Neuberger [34]. Consider the process of bonds owned $X(s)$ and the process of shares owned $Y(s)$. Let the trading strategy $\big(L(s), M(s)\big)$ be a pair of nondecreasing processes with $L(t) = M(t) = 0$, which are interpreted as the cumulative transfers, measured in shares of stock. $L(s)$ is measured in shares from bond to stock and $M(s)$ is measured in shares from stock to bond. Let $\kappa \in (0, 1)$ be the proportional transaction cost. The processes $X(s)$ and $Y(s)$ start with the initial values x and y, $s \in [t, T]$ and evolve according to

$$X(s) = x - \int_t^s S(\tau)(1+\kappa)\,dL(\tau) + \int_t^s S(\tau)(1-\kappa)\,dM(\tau) \tag{2.10}$$

and

$$Y(s) = y + L(s) - M(s). \tag{2.11}$$

The first integral in (2.10) represents buying shares of stock at a price increased by the proportional transaction cost, the second integral represents

selling stock at a reduced price of the transaction cost. In (2.11) we add the amount of the stocks bought and subtract the amount for the stocks sold to the initial amount of stocks owned.
According to the utility maximization approach of Hodges and Neuberger in [34], the price of a European Call option can be obtained as the difference between the maximum utility of the terminal wealth when there is no option liability and when there is such a liability. Following this approach, Barles and Soner consider two optimization problems. Let the *exponential utility function* be

$$U(\xi) = 1 - e^{-\gamma\xi},\ \xi \in \mathbb{R},$$

where $\gamma > 0$ is the *risk aversion factor*. The first value function is the expected utility from the final wealth without any option liabilities taken over the transfer processes

$$V_1(x, y, S(t), t) := \sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + Y(T)S(T)\big)],$$

the second one is the expected utility from the final wealth assuming that we have sold N European call options taken over the transfer processes

$$V_2(x, y, S(t), t) := \sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + Y(T)S(T) - N(S(T) - K)^+\big)].$$

Hodges and Neuberger postulate that the price of each option is equal to the maximal solution Λ of the algebraic equation

$$\begin{aligned} V_2(x + N\Lambda, y, S(t), t) = &\sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + N\Lambda + Y(T)S(T) \\ &- N(S(T) - K)^+\big)] \\ = &\sup_{L(\cdot),M(\cdot)} E[U\big(X(T) + Y(T)S(T)\big)] \\ = &V_1(x, y, S(t), t), \end{aligned}$$

which means that the option price Λ equals the increment of the initial capital at time t that is needed to cope with the option liabilities arising at T. By a linearity argument selling N options with risk aversion factor of γ yields the same price as selling one option with risk aversion factor γN. This leads to performing an asymptotic analysis as $\gamma N \to \infty$. Hence, we consider

$$U(\xi) = 1 - e^{-\gamma N\xi}$$

and

$$\varepsilon = \frac{1}{\gamma N}.$$

Then, we have

$$U_\varepsilon(\xi) = 1 - e^{-\frac{\xi}{\varepsilon}},\ \xi \in \mathbb{R}.$$

Our optimization problems become

$$V_1(x, y, S(t), t) = 1 - \inf_{L(\cdot),M(\cdot)} E[e^{-\frac{1}{\varepsilon}(X(T)+Y(T)S(T))}]$$

and

$$V_2(x,y,S(t),t) = 1 - \inf_{L(\cdot),M(\cdot)} E[e^{-\frac{1}{\varepsilon}(X(T)+Y(T)S(T)-(S(T)-K)^+)}].$$

For analysis simplification Barles and Soner define $z_{1,2} : \mathbb{R} \times (0,\infty) \times (0,T) \to \mathbb{R}$ by

$$V_1(x,y,S(t),t) = 1 - e^{-\frac{1}{\varepsilon}\left(x+yS(t)-z_1(y,S(t),t)\right)}$$

and

$$V_2(x,y,S(t),t) = 1 - e^{-\frac{1}{\varepsilon}\left(x+yS(t)-z_2(y,S(t),t)\right)}.$$

Then

$$z_1(y,S(t),T) = 0 \quad \text{and} \quad z_2(y,S(t),T) = (S(T)-K)^+$$

and the option price

$$\Lambda(x,y,S(t),t;\frac{1}{\varepsilon},1) = z_2(y,S(t),t) - z_1(y,S(t),t).$$

By the theory of stochastic optimal control [26], Barles and Soner state that the value functions V_1 and V_2 are the unique solutions of the dynamic programming equation

$$\min\{-V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S, -V_y + S(1+\kappa)V_x, V_y - S(1-\kappa)V_x\} = 0,$$

which leads to a dynamic programming equation for z_1 and z_2, which are independent of the variable x.
Supposing that the proportional transaction cost κ is equal to $a\sqrt{\varepsilon}$ for some constant $a > 0$, they prove that as $\varepsilon \to 0$ and $\kappa \to 0$

$$z_1 \to 0 \quad \text{and} \quad z_2 \to V,$$

where V is the unique (viscosity) solution of the nonlinear Black-Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2 S^2 V_{SS}^2 + rSV_S - rV,$$

where

$$\widetilde{\sigma}^2 = \sigma^2\bigg(1 + \Psi(e^{r(T-t)}a^2 S^2 V_{SS})\bigg). \tag{2.12}$$

Here σ denotes the historical volatility, $a = \kappa/\sqrt{\varepsilon}$ and $\Psi(x)$ is the solution to the following nonlinear ordinary differential equation (ODE)

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)} - x}, \; x \neq 0, \tag{2.13a}$$

with the initial condition

$$\Psi(0) = 0. \tag{2.13b}$$

The analysis of this ODE (2.13) by Barles and Soner in [4] implies that

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \Psi(x) = -1. \tag{2.14}$$

The property (2.14) encourages to treat the function $\Psi(\cdot)$ as the identity for large arguments and therefore to simplify the calculations. In this case the volatility becomes

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2S^2V_{SS}). \tag{2.15}$$

The existence of a viscosity solution to (1.2) for European options with the volatility given by (2.12) is proved by Barles and Soner in [4] and their numerical results indicate an economically significant price difference between the standard Black-Scholes model and the nonlinear model with transaction costs.

2.7 Risk Adjusted Pricing Methodology

In this model, proposed by Kratka in [46] and improved by Jandačka and Ševčovič in [41], the optimal time-lag δt between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected with the Risk Adjusted Pricing Methodology (RAPM) and the *modified volatility* is now of the form

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3(\frac{C^2M}{2\pi}SV_{SS})^{\frac{1}{3}}\right), \tag{2.16}$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ the risk premium measure.

It is worth mentioning that these nonlinear transaction cost models that we described above are all consistent with the linear model if the additional parameters for transaction costs are equal to zero and vanish (Le, $\Psi(\cdot)$, M). We will concentrate on four of the above mentioned models: Leland's model (2.8), the model of Barles and Soner (2.12), the identity model (2.15) and the Risk Adjusted Pricing Methodology (2.16). We will study these models – more precisely equations (1.2) and (1.3) where the volatility is given by the equations (2.8), (2.12), (2.15) and (2.16) – for both European and American Call options.
In general, an exact analytical solution leading to a closed expression is not known neither for European nor for American options in a market with transaction costs. In the next chapter we will analytically approach

the solution of (1.2) and (1.3) by a transformation, that facilitates the numerical solution. We will compare different approaches of solving this problem numerically by standard and compact finite-difference schemes in the chapter thereafter.

Chapter 3

Analytical Approach

Both the equations (1.2) and (1.3) are backward in time. In order to ease the numerical solution of (1.2) for the European Call option, we transform the problem into a forward parabolic problem. For the American Call option, we transform the original spatial domain of (1.3) subject to (1.7) into a fixed (unbounded) domain additionally to the forward transformation in time. Hence, in case of the American Call option, the domain does not depend on the free boundary $S_f(t)$ anymore and we simply calculate an algebraic constraint equation for the position of the free boundary.

3.1 Transformation of the European Call

In order to be able to solve the problem (1.2) with the corresponding volatilities subject to (1.5) numerically, we perform the following *variable transformation* (see e.g. [20, 74]):

$$x = \ln\left(\frac{S}{K}\right), \quad \tau = \frac{1}{2}\sigma^2(T-t), \quad u(x,\tau) = e^{-x}\frac{V(S,t)}{K}.$$

Since $S = Ke^x$ and $V = uS$, differentiation yields:

$$\begin{aligned} V_t &= u_\tau \tau_t S = -\frac{1}{2}\sigma^2 S u_\tau, \\ V_S &= u_x x_S S + u = u_x + u, \\ V_{SS} &= u_{xx} x_S + u_x x_S = \frac{1}{S}(u_{xx} + u_x). \end{aligned}$$

Plugging these derivatives into (1.2) leads to

$$0 = -\frac{1}{2}\sigma^2 S u_\tau + \frac{1}{2}\widetilde{\sigma}^2 S(u_{xx} + u_x) + rS(u_x + u) - ruS,$$

and a final multiplication by $-\frac{2}{S\sigma^2}$ gives

$$0 = u_\tau - \frac{\widetilde{\sigma}^2}{\sigma^2}(u_{xx} + u_x) - Du_x, \tag{3.1}$$

where $D = \frac{2r}{\sigma^2}$ and $\widetilde{\sigma}^2$ depends on the volatility model, $x \in \mathbb{R}$ and $0 \le \tau \le \widetilde{T} = \frac{\sigma^2 T}{2}$. Leland's model (2.8) becomes

$$\widetilde{\sigma}^2 = \sigma^2 \bigg(1 + Le\, \mathrm{sign}(u_{xx} + u_x) \bigg), \tag{3.2a}$$

Barles' and Soner's model (2.12)

$$\widetilde{\sigma}^2 = \sigma^2 \bigg(1 + \Psi\big(e^{\frac{2r\tau}{\sigma^2}} a^2 K e^x (u_{xx} + u_x) \big) \bigg), \tag{3.2b}$$

the identity model (2.15)

$$\widetilde{\sigma}^2 = \sigma^2 \bigg(1 + e^{\frac{2r\tau}{\sigma^2}} a^2 K e^x (u_{xx} + u_x) \bigg) \tag{3.2c}$$

and the Risk Adjusted Pricing Methodology (RAPM) (2.16) becomes

$$\widetilde{\sigma}^2 = \sigma^2 \bigg(1 + 3 \big(\frac{C^2 M}{2\pi} (u_{xx} + u_x) \big)^{\frac{1}{3}} \bigg). \tag{3.2d}$$

Now $u(x, \tau)$ solves (3.1) on the transformed domain $x \in \mathbb{R}$, $0 \le \tau \le \widetilde{T}$ subject to the following initial and boundary conditions resulting from (1.5):

$$\begin{aligned} u(x,0) &= (1 - e^{-x})^+ && \text{for } x \in \mathbb{R}, \\ u(x,\tau) &= 0 && \text{as } x \to -\infty, \\ u(x,\tau) &\sim 1 - e^{-D\tau - x} && \text{as } x \to \infty. \end{aligned} \tag{3.3}$$

Therefore the original problem (1.2) with (1.5) and the corresponding volatilities transforms to a forward problem (3.1) with the corresponding volatilities (3.2) subject to (3.3).
The numerical solution for the European Call option is specified in the first part of the next chapter. Once the problem is solved, u can easily be transformed into $V = Su$ and we obtain the price of the European Call option in the presence of transaction costs. Next, we address the transformation in case of the American Call option.

3.2 Transformation of the American Call

The purpose of converting the free-boundary problem for the nonlinear Black-Scholes equation (1.3) subject to (1.7) into a fully nonlinear parabolic equation defined on a fixed domain is the minimization of the error resulting from the discontinuity of V_{SS} at the free boundary. This is achieved by only considering the domain where (1.3) holds [78]. Following the idea of Ševčovič [72] we change the variables to:

$$\tau = T - t, \quad x = \ln\left(\frac{\varrho(\tau)}{S} \right) \Leftrightarrow S = e^{-x} \varrho(\tau), \quad \varrho(\tau) = S_f(T - \tau),$$

so that $x \in \mathbb{R}^+$ and $\tau \in [0, T]$. We then construct a portfolio

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t)$$

by buying $\Delta = V_S$ shares S and selling an option V. Differentiating Π with respect to x and τ gives us

$$\Pi_x = V_S S_x - S_x V_S - SV_{SS} S_x = S^2 V_{SS}$$

and

$$\begin{aligned}
\Pi_\tau &= V_S S_\tau + V_t t_\tau - S_\tau V_S - S(V_{SS} S_\tau + V_{St} t_\tau) \\
&= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)} S^2 V_{SS} + SV_{St} \\
&= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)} \Pi_x - S\partial_S(-V_t).
\end{aligned} \tag{3.4}$$

Substituting

$$-V_t = \frac{\widetilde{\sigma}^2}{2} S^2 V_{SS} - r(V - SV_S) - qSV_S = \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S$$

from (1.3) into (3.4) and using the fact that $-S\partial_S = \partial_x$, we get

$$\begin{aligned}
\Pi_\tau &= \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S - \frac{\varrho'(\tau)}{\varrho(\tau)}\Pi_x + \partial_x\left(\frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi\right) + S\partial_S(qSV_S) \\
&= \left(\frac{\widetilde{\sigma}^2}{2} - \frac{\varrho'(\tau)}{\varrho(\tau)} - r + q\right)\Pi_x - r\Pi + \frac{1}{2}\partial_x(\widetilde{\sigma}^2 \Pi_x).
\end{aligned}$$

We therefore obtain

$$0 = \Pi_\tau + \big(b(\tau) - \frac{\widetilde{\sigma}^2}{2}\big)\Pi_x - \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x) + r\Pi, \tag{3.5}$$

defined on $x \in \mathbb{R}^+$, $0 \le \tau \le T$, where

$$b(\tau) = \frac{\varrho'}{\varrho}(\tau) + r - q.$$

The terminal condition from (1.7) in the original variables (S, T) becomes the initial condition in the new variables $(x, 0)$:

$$\begin{aligned}
\Pi(x,0) &= V(S,T) - SV_S(S,T) \\
&= \begin{cases} -K & \text{for } S > K \Leftrightarrow x < \ln \frac{\varrho(0)}{K} \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{3.6a}$$

and the boundary conditions from (1.7) transform to

$$\Pi(x,\tau) = 0 \qquad \text{as } x \to \infty,\ 0 \le \tau \le T, \tag{3.6b}$$

$$\Pi(0,\tau) = -K \qquad \text{for } 0 \le \tau \le T. \tag{3.6c}$$

To complete the system of equations that enables the computation of the portfolio Π we need to use the last two conditions of (1.7) to obtain an expression at the free boundary position $\varrho(\tau)$. Differentiating and evaluating V at the free boundary gives us

$$V_S(S_f(t),t)S_f'(t) + V_t(S_f(t),t) = S_f'(t).$$

Using (1.7), we conclude that

$$V_t(S_f(t),t) = 0 \text{ for } 0 \leq \tau \leq T.$$

Computing (1.3) at the point $(S_f(t),t)$ or at $(0,\tau)$ in the transformed variables gives:

$$\begin{aligned} 0 &= V_t(S_f(t),t) + \frac{1}{2}\widetilde{\sigma}^2\Pi_x(0,\tau) + (r-q)S_f(t)V_S(S_f(t),t) - rV(S_f(t),t) \\ &= \frac{1}{2}\widetilde{\sigma}^2\Pi_x(0,\tau) + rK - q\varrho(\tau). \end{aligned}$$

As we have already assumed in Section 1.4.3, we have $r \geq q$ and therefore we obtain the last condition:

$$\varrho(\tau) = \frac{1}{2q}\widetilde{\sigma}^2\Pi_x(0,\tau) + \frac{rK}{q} \quad \text{with} \quad \varrho(0) = \frac{rK}{q}, \tag{3.6d}$$

where $0 \leq \tau \leq T$ and $\widetilde{\sigma}^2$ depends on the volatility model we choose. The volatility (2.8) from the Leland model becomes

$$\widetilde{\sigma}^2 = \sigma^2\Big(1 + Le\,\mathrm{sign}(\Pi_x)\Big), \tag{3.7a}$$

for (2.12) we get

$$\widetilde{\sigma}^2 = \sigma^2(1 + \Psi(e^{r\tau}a^2\Pi_x)), \tag{3.7b}$$

for (2.15) we obtain

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r\tau}a^2\Pi_x) \tag{3.7c}$$

and for (2.16)

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3\big(\frac{C^2M}{2\pi}\Pi_x\varrho(\tau)e^{-x}\big)^{\frac{1}{3}}\right). \tag{3.7d}$$

This transformed problem (3.5) subject to (3.6) with the corresponding volatilities (3.7) is solved by the split-step finite-difference method proposed by Ševčovič in [72] and elaborated on in Section 4.2 of this thesis.
Once we have numerically solved the transformed problem by calculating the solution to our portfolio $\Pi(x,\tau)$ and the free boundary $\varrho(\tau)$, we calculate the value of the American Call $V(S,t)$ option by transforming

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t)$$

back to the original variables. Since we know that

$$\frac{\Pi(x,\tau)}{S^2} = \frac{V(S,t)}{S^2} - \frac{V_S(S,t)}{S} = \partial_S\Big(-\frac{V(S,t)}{S}\Big),$$

we integrate the above equation from S to $S_f(t)$, take into account the boundary condition $V(S_f(t),t) = S_f(t) - K$ and obtain:

$$\begin{aligned}
\int_S^{S_f(t)} \frac{\Pi(\ln(\varrho(\tau)/S),\tau)}{S^2} dS &= \int_S^{S_f(t)} \partial_S\Big(-\frac{V(S,t)}{S}\Big) dS \\
\int_{\ln\frac{\varrho(\tau)}{S}}^{\ln\frac{\varrho(\tau)}{S_f(t)}} \frac{\Pi(x,\tau)}{S^2}(-S)dx &= -\frac{V(S_f(t),t)}{S_f(t)} + \frac{V(S,t)}{S} \\
S\int_0^{\ln\frac{\varrho(\tau)}{S}} \frac{\Pi(x,\tau)}{e^{-x}\varrho(\tau)}dx &= -S\frac{\varrho(\tau)-K}{\varrho(\tau)} + V(S,t)
\end{aligned}$$

$$V(S,T-\tau) = \frac{S}{\varrho(\tau)}\Big(\varrho(\tau) - K + \int_0^{\ln\frac{\varrho(\tau)}{S}} e^x\Pi(x,\tau)dx\Big). \qquad (3.8)$$

Therefore, (3.8) yields the price of the American Call option $V(S,t)$ in the presence and absence of transaction costs.

Chapter 4

Numerical Solution

Due to the lack of general closed–form solutions to the Black–Scholes equations, there are various numerical methods for solving Black–Scholes equations for European and American options.
For European Call and Put options, the Black–Scholes formulae (C.1) and (C.2) provide the correct answer, but for more complicated contracts in more general settings analytical formulae are seldom available and numerical methods have to be used to solve the problem. These vary from lattice methods (including binomial and trinomial approximations [15]), Monte-Carlo methods using the least-square techniques [40], analytical approximations [5, 11, 49], finite-element discretizations [31, 43] to finite-difference methods [2, 10, 14].
There are numerous other methods for pricing American options including the method of lines [52], front-tracking algorithms [76], penalty methods [79] and many others. One of the standard approaches for solving the Black–Scholes equation for American options consists of the transformation of the original equation into the heat equation posed on a semi–unbounded domain with a free boundary $S_f(t)$ [61, 75]. For a new alternative direct method using the Mellin transformation we refer to [42, 55].
Up to now, an exact analytical formula for the free boundary profile $S_f(t)$ in (1.3) subject to (1.7) is not known, but several authors derived approximate expressions to evaluate American Call and Put options in the linear case [30]. Recently, in a promising approach [71], Ševčovič obtained a semi–explicit formula for an American Call in the case of $r > q$. By transforming the linear Black–Scholes equation for the American Call option into a nonlinear parabolic equation on a fixed domain and applying Fourier sine and cosine transformations, he derives a nonlinear singular integral equation determining the shape of the free boundary. This integral equation can be solved effectively by the means of successive iterations.
Another standard method consists of the reformulation of the free boundary problem into a linear complementary problem (LCP) and the solution by the *Projected Successive Over Relaxation (PSOR)* method of Cryer [16]. Alternatively, penalty and front–fixing methods are developed (e.g. in [27, 53]). A disadvantage of these methods is the change of the underlying model.

A different approach [36] is based on a recursive calculation of the early exercise boundary, estimating the boundary only at some points and then approximating the whole boundary by Richardson extrapolation. Explicit boundary tracking algorithms are e.g. a *finite-difference bisection scheme* [45] or the *front–tracking strategy* of Han and Wu [32].
This work's emphasis is on finite-difference schemes, thus other methods will not be further elaborated on here. For more information on numerical methods we refer to [56, 57, 77] and the references therein.

4.1 European Call option

In this section we want to use finite-difference schemes to solve the transformed problem from Section 3.1

$$0 = u_\tau - \frac{\widetilde{\sigma}^2}{\sigma^2}(u_{xx} + u_x) - Du_x, \quad x \in \mathbb{R}, \quad 0 \le \tau \le \widetilde{T} \tag{4.1}$$

with the corresponding volatilities (3.2) subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= (1 - e^{-x})^+ && \text{for } x \in \mathbb{R}, \\ u(x,\tau) &= 0 && \text{as } x \to -\infty, \\ u(x,\tau) &\sim 1 - e^{-D\tau - x} && \text{as } x \to \infty. \end{aligned} \tag{4.2}$$

Hence, we first introduce the reader to finite-difference schemes and then present some numerical results.

4.1.1 Finite-Difference Schemes

The idea of finite-difference schemes is the approximation of the derivatives in (4.1) by difference quotients and the solution of the resulting discrete schemes.
For this purpose, we start by discretizing the domain of the transformed problem (4.1) with the corresponding volatilities (3.2) subject to (4.2). We then substitute the derivatives by appropriate difference quotients and address the nonconstant volatilities. We continue by examining the resulting schemes in terms of convergence, introduce both *classical* and *compact finite-difference schemes* for the European Call option and finally compare them to each other.

4.1.1.1 Grid

We begin by replacing $x \in \mathbb{R}$ and $\tau \in [0, \widetilde{T}]$ by a bounded interval $x \in [-R, R]$, $R > 0$. We discretize the new computational domain by a uniform grid (x_i, τ_n) with $x_i = ih$ and $\tau_n = nk$, where $h > 0$ denotes the spatial step, $k > 0$ is the time step, $i \in [-N, N]$, $-R = -Nh$, $R = Nh$, $n \in [0, M]$

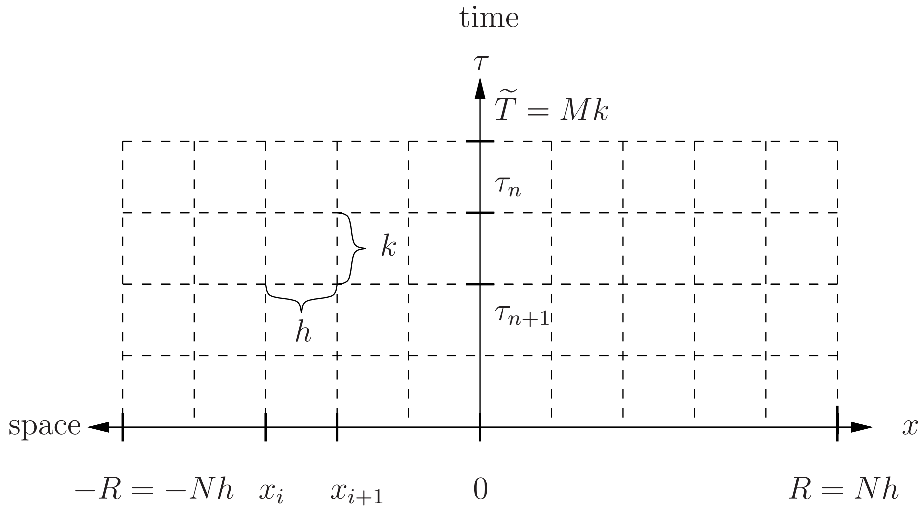


Figure 4.1: Uniform grid for a European Call option.

and $\widetilde{T} = Mk$ (see Figure 4.1).
We denote the approximate solution of (4.1) in x_i at time τ_n by $U_i^n \approx u(x_i, \tau_n)$ and discretize the initial and boundary conditions (4.2) in the following way:

$$\begin{aligned} U_i^0 &= (1 - e^{-ih})^+, \\ U_{-N}^n &= 0, \\ U_N^n &= 1 - e^{-Dnk-Nh}. \end{aligned} \tag{4.3}$$

For a more appropriate treatment of the unbounded spatial domain $x \in \mathbb{R}$ so-called *artificial boundary conditions* [24] can be introduced to confine the unbounded domain of (4.1) to a bounded computational domain. This new approach will be an issue of a subsequent paper.

4.1.1.2 Difference Quotients

The spatial derivative can be approximated with **forward differences**:

$$u_x(x, \tau) = \frac{u(x+h, \tau) - u(x, \tau)}{h} + \mathcal{O}(h)$$

or with **backward differences**:

$$u_x(x, \tau) = \frac{u(x, \tau) - u(x-h, \tau)}{h} + \mathcal{O}(h).$$

The sum of these differences results in **central differences** and we get

$$u_x(x, \tau) = \frac{u(x+h, \tau) - u(x-h, \tau)}{2h} + \mathcal{O}(h^2).$$

For the second spatial derivative we compute with the Taylor formula

$$u_{xx}(x_i, \tau_n) = \frac{u(x+h, \tau) - 2u(x, \tau) + u(x-h, \tau)}{h^2} + \mathcal{O}(h^2).$$

We recall that a function $f(h)$ is in $\mathcal{O}(h^r)$, if there exists a constant $M > 0$, such that $|f(h)| \leq M|h^r|$ as $h \to 0$, meaning that the quantity $f(h)$ is bounded by a constant multiple of h^r for sufficiently small h (cf. [65]). We call the error between differential quotient and difference quotient the *truncation error*.

To discretize (4.1) we introduce the following notation for the forward difference quotient with the spatial step size h:

$$D_h^+ U_i^n := \frac{U_{i+1}^n - U_i^n}{h} \approx u_x(x_i, \tau_n),$$

where we leave out the error term $\mathcal{O}(h)$. Similarly, the backward difference quotient with respect to the spatial variable is denoted as

$$D_h^- U_i^n := \frac{U_i^n - U_{i-1}^n}{h} \approx u_x(x_i, \tau_n)$$

and the central difference quotient as

$$D_h^0 U_i^n := \frac{U_{i+1}^n - U_{i-1}^n}{2h} \approx u_x(x_i, \tau_n)$$

omitting the truncation error of $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$. For the second spatial derivative we introduce the standard difference quotient

$$D_h^2 U_i^n := \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} \approx u_{xx}(x_i, \tau_n),$$

with the error term $\mathcal{O}(h^2)$. Note that central differences in the time variable are never used in practice because they always lead to bad numerical schemes, that are inherently unstable (see [74]).

Most of the resulting schemes lead to systems of equations that can be written in matrix form:

$$A^n U^{n+1} = B^n U^n + d^n, \tag{4.4}$$

where

$$U^n = \left(U_{-N+1}^n, \cdots, U_0^n, \cdots, U_{N-1}^n\right)^\top \in \mathbb{R}^{2N-1},$$

$$A^n = \begin{pmatrix} a_0 & a_1 & 0 & \cdots & 0 \\ a_{-1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_{-1} & a_0 \end{pmatrix} \in \mathbb{R}^{(2N-1)\times(2N-1)},$$

$$B^n = \begin{pmatrix} b_0 & b_1 & b_2 & 0 & \cdots & 0 \\ b_{-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_{-2} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & b_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & b_1 \\ 0 & \cdots & 0 & b_{-2} & b_{-1} & b_0 \end{pmatrix} \in \mathbb{R}^{(2N-1)\times(2N-1)}$$

and

$$d^n = \begin{pmatrix} b_{-2}U^n_{-N-1} + b_{-1}U^n_{-N} - a_{-1}U^{n+1}_{-N} \\ b_{-2}U^n_{-N} \\ 0 \\ \vdots \\ 0 \\ b_2U^n_N \\ b_1U^n_N + b_2U^n_{N+1} - a_1U^{n+1}_N \end{pmatrix} \in \mathbb{R}^{2N-1}.$$

The matrix A^n is triagonal, so that the resulting systems can be solved with linear effort $\mathcal{O}(N)$ using the Thomas algorithm [70, 32]. This is done by first decomposing the matrix $A^n = L^nR^n$ into a lower and an upper bidiagonal matrix and secondly solving $L^nR^nU^{n+1} = B^nU^n + d^n$ by forward and backward substitution. Hence, we first solve $L^nY^n = B^nU^n + d^n$ for Y^n and then we solve $R^nU^{n+1} = Y^n$ for U^{n+1}.
The schemes we will consider are *two-level schemes*, meaning that they only involve U at two time levels n and $n+1$. We will introduce both *three-point* and *five-point schemes*, meaning that they involve U at three and five spatial levels. In case we use a five-point approximation in the spatial variable, that is $b_{-2}, b_2 \neq 0$, the vector d^n involves U^n_{-N-1} and U^n_{N+1}, which are outside the area we are considering. We will impose the conditions

$$U^n_{-N-1} = 0 \quad \text{and} \quad U^n_{N+1} = 1 - e^{-Dnk-(N+1)h} \tag{4.5}$$

for these ghost or *auxiliary values* (see [65]). We further suppose that

$$\sum_{i=-1}^{1} a_i = \sum_{i=-2}^{2} b_i = 1,$$

which is satisfied by any consistent scheme after normalization of the coefficients (cf. [59]).
We bear in mind that we say a scheme is of order a in time and b in space, abbreviated by (a, b), if its truncation error is of order $\mathcal{O}(k^a + h^b)$.

Remark 4.1. *A word of caution needs to be said about the accuracy when approximating the derivatives with finite differences. Taylor's expansion assumes the existence of several derivatives for $u(x, \tau)$. However, most option pricing problems have nonsmooth pay-offs and therefore discontinuous derivatives at the strike price, which is equivalent to nonsmooth initial data for our transformed problem at $u(0, 0)$ (see Figure 4.2).*
*In [54] this problem is overcome and the accuracy is improved by a **grid stretching** technique, which is based on the idea of placing more points in the neighborhood of the grid points where the non-differentiable condition occurs.*
We will keep this smoothing strategy in mind, but for the sake of the simplicity of the presentation we use the grid that we described earlier in this section.

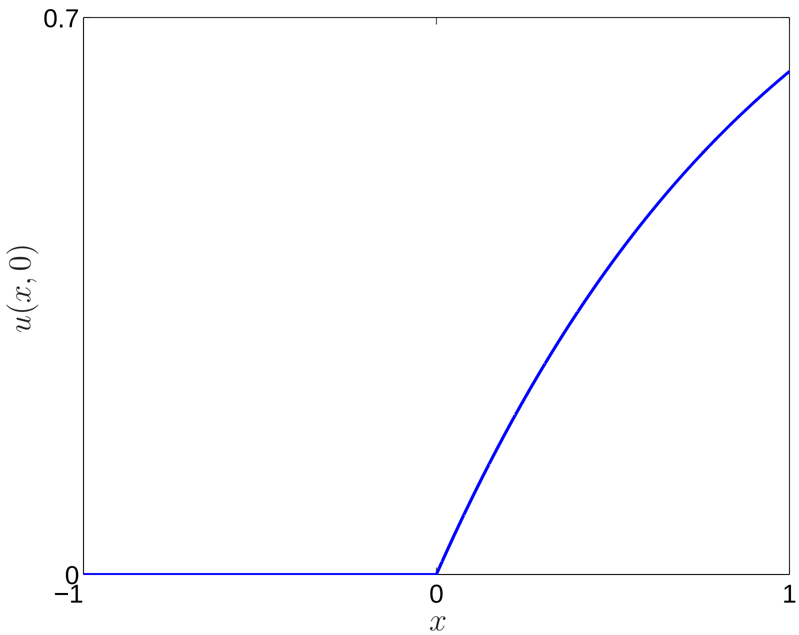


Figure 4.2: Nonsmooth initial data (4.3) for the transformed problem (4.1) at $u(0,0)$.

4.1.1.3 Volatility Functions

There are different ways of treating the derivatives in the volatility. The modified volatilities (3.2) can all be written in the form

$$\widetilde{\sigma}^2 = \sigma^2\big(1 + s(x,\tau)\big),$$

where $s(x,\tau)$ denotes the *volatility correction* in x at time τ, which depends on the first and second spatial derivatives of u.
Düring [21] suggests a smoother approximation of u_{xx} for the volatility correction by choosing:

$$u_{xx}(x_i,\tau_n) \approx \frac{U^n_{i+2} - 2U^n_i + U^n_{i-2}}{4h^2} := D^2_{2h}U^n_i,$$

with the truncation error of order $\mathcal{O}(h^2)$. We will follow Düring's suggestion and treat the nonlinearity explicitly in all the schemes. Therefore we denote the volatility correction in x_i at time τ_n for Leland's model with the volatility (3.2a) by

$$s^n_i = \sqrt{\frac{2}{\pi}}\frac{\kappa}{\sigma\sqrt{\delta t}}\,\mathrm{sign}\left(D^2_{2h}U^n_i + D^0_hU^n_i\right), \tag{4.6a}$$

the volatility correction for Barles' and Soner's model with the volatility (3.2b) by

$$s^n_i = \Psi\Big(e^{D\tau_n + x_i}a^2K(D^2_{2h}U^n_i + D^0_hU^n_i)\Big), \tag{4.6b}$$

the volatility correction in case of treating $\Psi(\cdot)$ as the identity with the volatility (3.2c) by

$$s_i^n = e^{D\tau_n + x_i} a^2 K \big(D_{2h}^2 U_i^n + D_h^0 U_i^n\big) \tag{4.6c}$$

and the volatility correction for the Risk Adjusted Pricing Methodology with the volatility (RAPM) (3.2d) by

$$s_i^n = 3\left(\frac{C^2 M}{2\pi}(D_{2h}^2 U_i^n + D_h^0 U_i^n)\right)^{\frac{1}{3}}. \tag{4.6d}$$

A problem occurring with s_i^n is the calculation at the boundary, since theoretically we need $U^n \in \mathbb{R}^{2N+3}$ to be able to calculate s_{N-1}^n and s_{-N+1}^n. This calculation involves U_{-N-1}^n and U_{N+1}^n, which are outside the computational domain. Düring states in [21] that the influence of the nonlinearity at the boundary is not significant and can be therefore neglected for large R. We will assume that (4.5) are valid and hence denote

$$s^n = \begin{pmatrix} s_{-N+1}^n, & \cdots, & s_0^n, & \cdots, & s_{N-1}^n \end{pmatrix}^\top \in \mathbb{R}^{2N-1}.$$

For the volatility model of Barles and Soner with the volatility (3.2b), the ordinary differential equation (2.13) has to be solved. We solve it with the `ode45` function in MATLAB, which is based on an explicit Runge-Kutta $(4, 5)$ one-step solver, the Dormand-Prince pair [18] (see Figure 4.3).

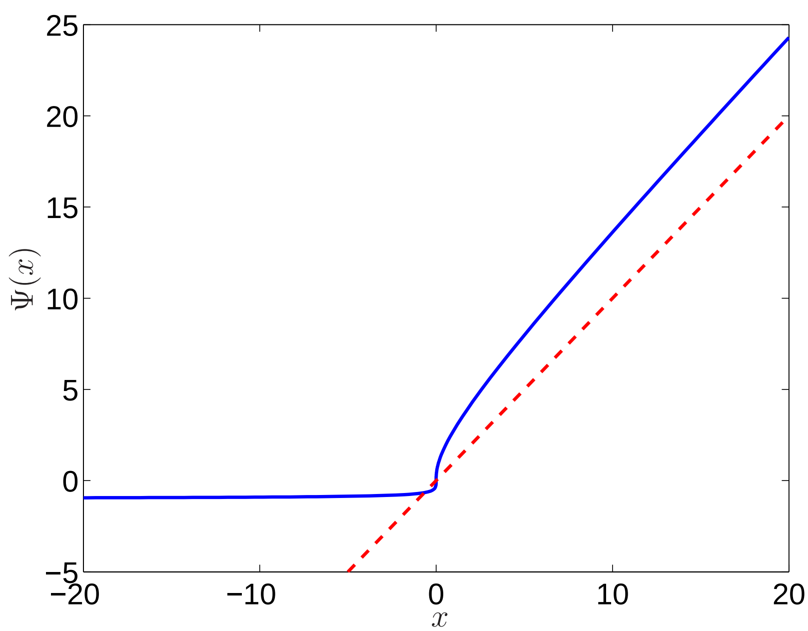


Figure 4.3: Solution Ψ to (2.13) using the `MATLAB` routine `ode45` (blue solid line) and the identity function $\Psi = x$ (red dotted line).

The values at the argument of $\Psi(\cdot)$ in the volatility correction for Barles's and Soner's model (4.6b) are obtained by a cubic spline interpolation between the values that were calculated by the MATLAB routine.

4.1.1.4 Existence and Convergence

In order to give a reasonable approximation for the sequence of the solutions to (4.4),

- a uniform solution U^n has to exist for each $n \in [0, M-1]$;
- and U_i^n has to converge towards the exact solution of (4.1) as $k \to 0$, $h \to 0$.

We first recall the terms of *existence* and *convergence* for the linear case, when the volatility correction s_i^n is equal to zero – and therefore $A^n := A$, $B^n := B$ and the coefficients in d^n are constant.
According to [39], a uniform solution to the system of equations (4.4) exists, if the matrix A is regular. The scheme (4.4) converges, if it is consistent and stable. Thus, we specify the terms *consistency* and *stability*.

- *Consistency:*
  A scheme $L_{h,k}$ of order (a, b) is consistent, if there exists a constant $M > 0$, such that

  $$\max_{i,n} |L_{h,k} u_i^n| \leq M(k^a + h^b)$$

  for sufficiently small $k, h > 0$. Here, u_i^n is the exact solution to (4.1) in (x_i, τ_n) and $L_{h,k}$ is the finite-difference scheme that omits the truncation error of order $\mathcal{O}(k^a + h^b)$ (such as (4.8), (4.11), etc.).

- *Stability:*
  For the stability of (4.4), we consider the computer-calculated vector including the rounding errors $\widehat{U}^n$. Then $A\widehat{U}^{n+1} = B\widehat{U}^n + d^n + r^n$, where r^n denotes the rounding errors. The error vector $e^n = \widehat{U}^n - U^n$ complies with $Ae^{n+1} = Be^n + r^n$. For simplicity, we assume that $e^0 \neq 0$, meaning that there is already a rounding error when evaluating the initial condition. At the same time, we assume that the matrix-vector multiplication to obtain U^{n+1} works accurately, hence $r^n = 0$ for $n \in [0, M-1]$. Therefore, we have the error evolution

  $$e^{n+1} = A^{-1}Be^n = (A^{-1}B)^2 e^{n-1} = \cdots = (A^{-1}B)^{n+1} e^0.$$

  In order to have a stable system, previous errors have to be damped and therefore we require $(A^{-1}B)^{n+1} e^0 \to 0$ as $n \to \infty$. According to Lemma 6.7 in [31], this is equivalent to the absolute value of the eigenvalues of $A^{-1}B$ being less than 1. Hence, in the linear case, a scheme is stable, if

  $$\rho(A^{-1}B) = \max\{|\zeta| : \zeta \text{ is eigenvalue of } A^{-1}B\} < 1.$$

When the above statements are true, a reasonable approximation to the linear system (4.4) can be found. Whether the solution to (4.4) contains small oscillations, depends on the *dissipation*. A scheme is called *non-oscillatory* or *dissipative*, if the eigenvalues ζ of $A^{-1}B$ satisfy an inequality of the form

$$|\zeta| \leq 1 - p|ih|^{2q},$$

when $|ih| \leq \pi$, for some real constant $p > 0$ and $q \in \mathbb{N}$. Note, that these oscillations are not the result of instability, but of inadequate resolution (cf. [65, 58]).

The above statements are valid for the linear case $s_i^n = 0$. In the nonlinear case, when $s_i^n \neq 0$, these terms are hard to state and prove for arbitrary schemes and arbitrary coefficients in A^n, B^n and d^n. One approach is to freeze the coefficients in (4.4) by assuming them to be constant at each point (x_i, τ_n) and check for stability. It is known that for linear parabolic problems with variable coefficients a mild strengthening of the *local* stability is sufficient to ensure overall stability [58]. For nonlinear problems, however, the limits of what can be generally proved are reached quickly. We will study the nonlinear case numerically in Section 4.1.2 and now introduce different finite-difference schemes for the European Call option and recall their properties corresponding to the linear case.

For the future we introduce the abbreviations

$$\lambda = -(1 + D), \quad \alpha = \frac{\lambda h}{2}, \quad r = \frac{k}{h^2}, \quad \mu = \frac{k}{h}.$$

4.1.1.5 Forward-Time Central-Space (explicit)

We remember that we are replacing the derivatives in the transformed Black–Scholes equation (4.1) for the European Call option by appropriate finite-difference quotients. Equation (4.1) can also be rearranged to:

$$u_\tau = (1 + s)(u_{xx} + u_x) + Du_x = s(u_{xx} + u_x) + (1 + D)u_x + u_{xx}, \qquad (4.7)$$

where s is the continuous volatility correction depending on the model. Now, (4.7) has the form of a convection-diffusion equation with a nonlinear term. In case of this explicit scheme, the time derivative is approximated by $D_k^+ U_i^n$, the first spatial derivative by $D_h^0 U_i^n$ and the second spatial derivative by both $D_h^2 U_i^n$ and $D_{2h}^2 U_i^n$, leaving the error terms of order $\mathcal{O}(k + h^2)$. Hence, this scheme is of order $(1, 2)$. Replacing all the derivatives of (4.7) by their corresponding finite-difference quotients we get:

$$D_k^+ U_i^n = s_i^n \left(D_{2h}^2 U_i^n + D_h^0 U_i^n \right) + (1 + D) D_h^0 U_i^n + D_h^2 U_i^n \qquad (4.8)$$

or

$$\begin{aligned}\frac{U_i^{n+1} - U_i^n}{k} &= s_i^n \left(\frac{U_{i+2}^n - 2U_i^n + U_{i-2}^n}{4h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h} \right) \\ &\quad + (1 + D) \frac{U_{i+1}^n - U_{i-1}^n}{2h} + \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}.\end{aligned}$$

This is equivalent to

$$\begin{aligned} U_i^{n+1} &= U_i^n + ks_i^n\left(\frac{U_{i+2}^n - 2U_i^n + U_{i-2}^n}{4h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h}\right) \\ &\quad +k(1+D)\frac{U_{i+1}^n - U_{i-1}^n}{2h} + k\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} \\ &= \frac{ks_i^n}{4h^2}U_{i-2}^n + \left(\frac{k}{h^2} - \frac{k(1+D+s_i^n)}{2h}\right)U_{i-1}^n + \left(1 - \frac{k(s_i^n+4)}{2h^2}\right)U_i^n \\ &\quad +\left(\frac{k(1+D+s_i^n)}{2h} + \frac{k}{h^2}\right)U_{i+1}^n + \frac{ks_i^n}{4h^2}U_{i+2}^n. \end{aligned}$$

Writing this scheme for $i \in [-N, N]$ resolves in the system of equations (4.4) with the matrix coefficients:

$$\begin{aligned} & & b_{-2} &= \tfrac{ks_i^n}{4h^2}, \\ a_{-1} &= 0, & b_{-1} &= \tfrac{k}{h^2} - \tfrac{k(1+D+s_i^n)}{2h}, \\ a_0 &= 1, & b_0 &= 1 - \tfrac{k(s_i^n+4)}{2h^2}, \\ a_1 &= 0, & b_1 &= \tfrac{k(1+D+s_i^n)}{2h} + \tfrac{k}{h^2}, \\ & & b_2 &= \tfrac{ks_i^n}{4h^2}, \end{aligned}$$

or in our notation

$$\begin{aligned} & & b_{-2} &= \tfrac{rs_i^n}{4}, \\ a_{-1} &= 0, & b_{-1} &= \tfrac{\mu(\lambda-s_i^n)}{2} + r, \\ a_0 &= 1, & b_0 &= 1 - \tfrac{r(s_i^n+4)}{2}, \\ a_1 &= 0, & b_1 &= \tfrac{\mu(s_i^n-\lambda)}{2} + r, \\ & & b_2 &= \tfrac{rs_i^n}{4}. \end{aligned}$$

According to the stability requirement in the linear case (see [65]), we need to have the necessary stability condition:

$$r \leq \frac{1}{2}. \tag{4.9}$$

The solution is non-oscillatory in the linear case if:

$$|\alpha| \leq 1. \tag{4.10}$$

4.1.1.6 Backward-Time Central-Space (implicit)

The backward Euler finite-difference scheme implies

$$D_k^- U_i^{n+1} = s_i^n\left(D_{2h}^2 U_i^n + D_h^0 U_i^n\right) + (1+D)D_h^0 U_i^{n+1} + D_h^2 U_i^{n+1}, \tag{4.11}$$

or

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{k} &= s_i^n\left(\frac{U_{i+2}^n - 2U_i^n + U_{i-2}^n}{4h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h}\right) \\ &\quad +(1+D)\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2h} + \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} \end{aligned}$$

with the error order $(1,2)$. Rearranging and grouping leads to the following matrix coefficients:

$$\begin{array}{rclrcl} & & & b_{-2} & = & \frac{ks_i^n}{4h^2}, \\ a_{-1} & = & \frac{k(1+D)}{2h} - \frac{k}{h^2}, & b_{-1} & = & -\frac{ks_i^n}{2h}, \\ a_0 & = & 1 + \frac{2k}{h^2}, & b_0 & = & 1 - \frac{ks_i^n}{2h^2}, \\ a_1 & = & -\frac{k(1+D)}{2h} - \frac{k}{h^2}, & b_1 & = & \frac{ks_i^n}{2h}, \\ & & & b_2 & = & \frac{ks_i^n}{4h^2}, \end{array}$$

or

$$\begin{array}{rclrcl} & & & b_{-2} & = & \frac{rs_i^n}{4}, \\ a_{-1} & = & \frac{\lambda\mu}{2} - r, & b_{-1} & = & -\frac{\mu s_i^n}{2}, \\ a_0 & = & 1 + \frac{2k}{h^2}, & b_0 & = & 1 - \frac{rs_i^n}{2}, \\ a_1 & = & -\frac{\lambda\mu}{2} - r, & b_1 & = & \frac{\mu s_i^n}{2}, \\ & & & b_2 & = & \frac{rs_i^n}{4}. \end{array}$$

According to [65] it is unconditionally stable and non-oscillatory in the linear case if (4.10) is satisfied.

4.1.1.7 Crank-Nicolson

This classical finite-difference scheme computes the solution better than the forward and backward difference methods due to its superior order of $(2,2)$ (cf. [61, 65]). To improve the order and the stability we average the forward- (4.8) and backward (4.11) difference method by summing them up. However, this time we exclusively approach the second spatial derivative by $D_h^2 U_i^n$ except in the nonlinear volatility term s_i^n. Replacing all the derivatives in (4.7) by their corresponding finite-difference quotients we get:

$$\begin{aligned} D_k^+ U_i^n + D_k^- U_i^{n+1} = {} & s_i^n\big(D_h^2 U_i^n + D_h^0 U_i^n\big) + s_i^n\big(D_h^2 U_i^{n+1} + D_h^0 U_i^{n+1}\big) \\ & + (1+D)\big(D_h^0 U_i^n + D_h^0 U_i^{n+1}\big) \\ & + D_h^2 U_i^n + D_h^2 U_i^{n+1}. \end{aligned} \tag{4.12}$$

This is equivalent to

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{k} = {} & \frac{s_i^n}{2}\left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h}\right) \\ & + \frac{s_i^n}{2}\left(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2h}\right) \\ & + (1+D)\frac{U_{i+1}^n - U_{i-1}^n + U_{i+1}^{n+1} - U_{i-1}^{n+1}}{4h} \\ & + \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n + U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{2h^2}. \end{aligned}$$

Rearranging leads to the linear system (4.4) with the following coefficients:

$$\begin{array}{rcl} a_{-1} & = & s_i^n(-\frac{r}{2} + \frac{\mu}{4}) - \frac{r}{2} - \frac{\lambda\mu}{4}, \\ a_0 & = & 1 + r(1 + s_i^n), \\ a_1 & = & s_i^n(-\frac{r}{2} - \frac{\mu}{4}) - \frac{r}{2} + \frac{\lambda\mu}{4}, \end{array}$$

$$\begin{aligned} b_{-1} &= s_i^n(\tfrac{r}{2} - \tfrac{\mu}{4}) + \tfrac{r}{2} + \tfrac{\lambda\mu}{4}, \\ b_0 &= 1 - r(1 + s_i^n), \\ b_1 &= s_i^n(\tfrac{r}{2} + \tfrac{\mu}{4}) + \tfrac{r}{2} - \tfrac{\lambda\mu}{4}. \end{aligned}$$

The Crank-Nicolson scheme is unconditionally stable in the linear case [65].

4.1.1.8 Rigal's Compact Schemes

Compact difference schemes differ from classical schemes because they improve the order of the scheme by eliminating lower order terms in the truncation error. Rigal [59] develops two-level three-point finite-difference schemes of order $(2, 4)$ that are stable and non-oscillatory and yield more efficient and accurate results than implicit fourth-order schemes. Düring follows Rigal's ideas and generalizes his results for a nonlinear Black–Scholes equation [21]. A general two-level three-point scheme for the problem (4.7) can be written as:

$$\begin{aligned} D_k^+ U_i^n = {} & (1 + s_i^n)\Big(\big(\frac{1}{2} + A_1\big) D_h^2 U_i^n + \big(\frac{1}{2} + B_1\big) D_h^0 U_i^n\Big) \\ & + (1 + s_i^n)\Big(\big(\frac{1}{2} + A_2\big) D_h^2 U_i^{n+1} + \big(\frac{1}{2} + B_2\big) D_h^0 U_i^{n+1}\Big) \\ & + D\big(\frac{1}{2} + B_1\big) D_h^0 U_i^n + D\big(\frac{1}{2} + B_2\big) D_h^0 U_i^{n+1}, \end{aligned} \tag{4.13}$$

where A_1, A_2, B_1 and B_2 are real constants which should be chosen in such a way that they eliminate the lower order terms in the truncation error. Note, that if these constants are equal to zero, then (4.13) reduces to the classical Crank-Nicolson scheme (4.12) of order $(2, 2)$. If we choose

$$\begin{aligned} B_1 &= \tfrac{1+4r^2\alpha^2}{12\beta r}, \\ B_2 &= -\tfrac{1+4r^2\alpha^2}{12\beta r}, \\ A_1 &= -\tfrac{1}{12k\beta}(-2h^2 + 6\widetilde{\lambda}^2 k^2 B_2 - k^2\widetilde{\lambda}^2 - 12k\beta^2 B_2), \\ A_2 &= -\tfrac{1}{12k\beta}(2h^2 + 6\widetilde{\lambda}^2 k^2 B_2 + k^2\widetilde{\lambda}^2 + 12k\beta^2 B_2), \end{aligned}$$

with $\beta := 1 + s_i^n$ and $\widetilde{\lambda} := -(1 + s_i^n + D)$, plug them into the equation (4.13) and rearrange the U_i^ns, then our coefficients become

$$\begin{aligned} a_{-1} &= -\tfrac{12r\beta^2 - 2\beta + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4 + 6r\widetilde{\lambda} h\beta - \widetilde{\lambda} h - r^2\widetilde{\lambda}^3 h^3}{24\beta}, \\ a_0 &= \tfrac{10\beta + 12r\beta^2 + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4}{12\beta}, \\ a_1 &= -\tfrac{12r\beta^2 - 2\beta + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4 - 6r\widetilde{\lambda} h\beta + \widetilde{\lambda} h + r^2\widetilde{\lambda}^3 h^3}{24\beta}, \\ b_{-1} &= \tfrac{12r\beta^2 + 2\beta + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4 + 6r\widetilde{\lambda} h\beta + \widetilde{\lambda} h + r^2\widetilde{\lambda}^3 h^3}{24\beta}, \\ b_0 &= \tfrac{-10\beta + 12r\beta^2 + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4}{12\beta}, \\ b_1 &= \tfrac{12r\beta^2 + 2\beta + r\widetilde{\lambda}^2 h^2 + r^3\widetilde{\lambda}^4 h^4 - 6r\widetilde{\lambda} h\beta - \widetilde{\lambda} h - r^2\widetilde{\lambda}^3 h^3}{24\beta}. \end{aligned}$$

This scheme is known as the *R3C scheme* [21]. Note that if $\beta = 1$ or $s_i^n = 0$ this scheme reduces to the *R3B scheme* developed by Rigal [59], which is also unconditionally stable and non-oscillatory in the linear case. It is proved in [22], that the R3C scheme is of order $(2, 4)$, unconditionally stable and non-oscillatory for the volatility model of Barles and Soner.

4.1.1.9 Algorithm

The following algorithm summarizes the calculation of the price $V(S,t)$ for a European Call option in the presence (or absence) of transaction costs by the above-described finite-difference schemes:

Algorithm 1 Computation of the price $V(S,t)$ for the European Call

Input parameters: R, T, h, k, M, N, r, K, D, σ, Le, a, C, M

solve the ODE (2.13) required for the volatility model of Barles and Soner and interpolate the solution
initialize U^0 according to (4.3) and transform U^0 into V^0
for $i = -N+1, .., N-1$ **do**
  $U_i^0 = \max(1 - \exp(-ih), 0)$
  $V_i^0 = U_i^0 K \exp(ih)$
end for
set $u = [0; U^0; 1 - \exp(-Nh)]$
set $v = [0; V^0; K(\exp(Nh) - 1)]$
calculate U^{n+1} at each time level
for $n = 0 : M-1$ **do**
  calculate the volatility correction s^n depending on the volatility model using U^n (in the case of Barles' and Soner's model use the interpolated solution of (2.13), in the case without transaction costs $s^n = (0, \cdots, 0)^\top \in \mathbb{R}^{2N-1}$)
  fill the matrices A^n and B^n and the vector d^n with the corresponding coefficients depending on the finite-difference scheme using s^n
  L-R-decompose $A^n = L^n R^n$
  solve $L^n y^n = B^n U^n + d^n$ for y^n
  solve $R^n U^{n+1} = y^n$ for U^{n+1}
  transform U^{n+1} into V^{n+1}
  save the solution in the array $u = \left[u \quad [0; U^{n+1}; 1 - \exp(-Nh - D(n+1)k)]\right]$
  save the solution in the original variables in the array $v = \left[v \quad [0; V^{n+1}; K(\exp(Nh) - \exp(-D(n+1)k))]\right]$
  start over with the loop over n
end for
plot v at each time level and each stock price

4.1.2 Comparison Study

In [20] the above described finite-difference schemes are compared to each other for the volatility model of Barles and Soner (4.6b). Conclusively, the

Crank-Nicolson (CN) and the R3C schemes are superior to the Forward-Time Central-Space and Backward-Time Central-Space schemes. We will therefore concentrate on the CN and the R3C scheme and compare the four different transaction cost models (4.6) calculated by those two schemes to each other and to the linear model without transaction costs.
For our calculations we use the following parameters:

$$r = 0.1, \quad \sigma = 0.2, \quad K = 100, \quad T = 1 \text{ (one year)},$$

$$R = 1, \quad k = 0.001, \quad h = 0.1.$$

Figure 4.4 shows the structure of the price for the European Call option without transaction costs, computed with the Crank-Nicolson scheme. The structure computed with the R3C scheme is similar, hence we abstain from including this figure.

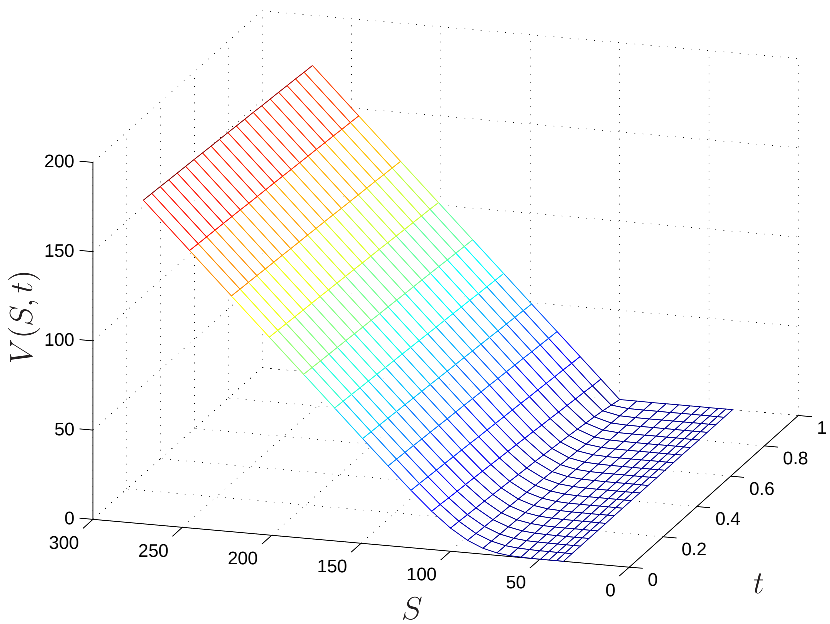
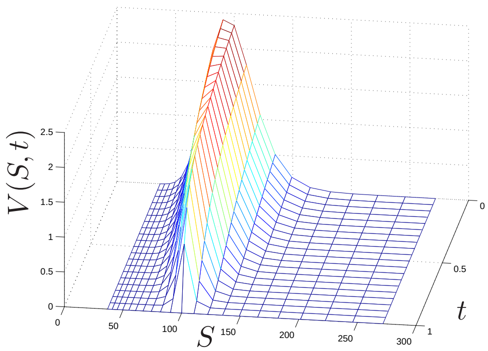


Figure 4.4: Value of a European Call option $V(S,t)$ in the absence of transaction costs computed with the Crank-Nicolson scheme.

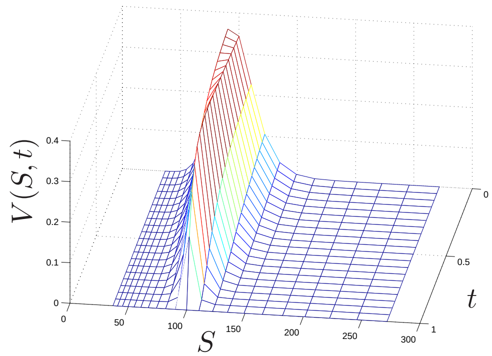
The influence of transaction costs modeled by the volatilities (2.8), (2.12), (2.15) and (2.16) and computed with the Crank-Nicolson finite-difference scheme can be seen in Figure 4.5. There, we plot the difference

$$V_{nonlinear}(S,t) - V_{linear}(S,t)$$

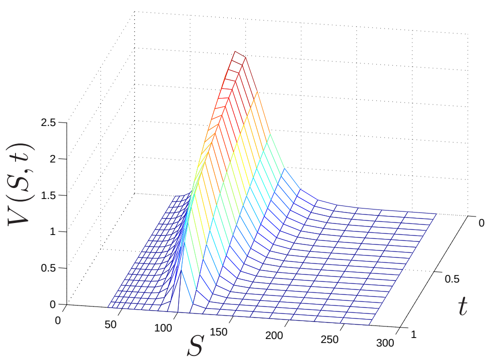
between the price of the European Call option with transaction costs and the price of the European Call without transaction costs for each model. As expected, the numerical results show an economically significant price



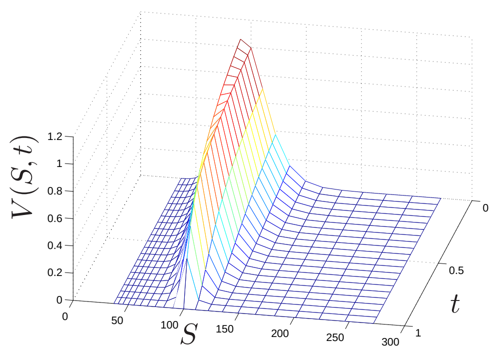
(a) Barles' and Soner's model ($a = 0.02$) vs. linear model.



(b) $\Psi(x) := x$ chosen as the identity ($a = 0.02$) vs. linear model.



(c) Leland's model ($\delta t = 0.01$, $\kappa = 0.05$) vs. linear model.



(d) RAPM model ($M = 0.01$, $C = 30$) vs. linear model.

Figure 4.5: The influence of transaction costs $V_{nonlinear}(S,t) - V_{linear}(S,t)$.

deviation between the standard (linear) Black–Scholes model and the nonlinear models.
The difference is not symmetric for all the transaction cost models, but decreases closer to the expiry date. This is an expected consequence of the decreasing necessity of portfolio adjustment and hence lower transaction costs closer to expiry. The difference is maximal at one year to expiry at $S \approx 95$, where the nonlinear price is significantly higher than the linear price. At this point with the given parameters Barles' and Soner's model provides the highest price (≈ 12.4), followed by Leland's model (≈ 11.9), Risk Adjusted Pricing Methodology (RAPM) (≈ 11.0), the identity (≈ 10.0) and finally the linear model with the constant volatility without transaction costs (≈ 9.9) (see Figure 4.6).
For each volatility model and each difference scheme we compare the error of accuracy of the above computation one year to expiry, that is at $t = 0$ or $\tau = \widetilde{T} = Mk$, and denote this ℓ^2-error by

$$err_2(Mk) = \left(h \sum_{i=-N}^{N} |u(x_i, \widetilde{T}) - U_i^M|^2 \right)^{\frac{1}{2}}.$$

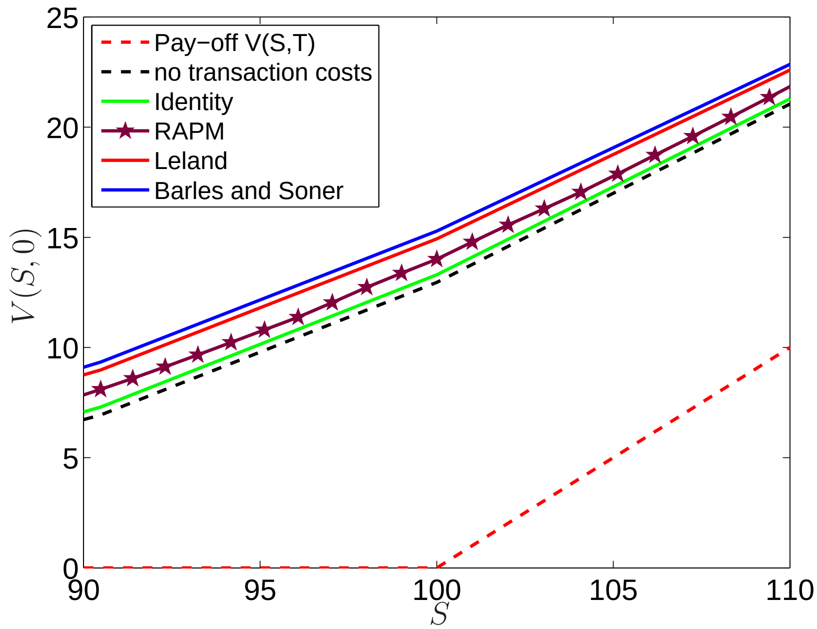


Figure 4.6: Price of a European Call option $V(S,0)$ for different transaction cost models vs. the price without transaction costs.

For the reference solution $u(x_i, \widetilde{T})$ we compute a solution for each model with the corresponding finite-difference scheme on a very fine grid with the step sizes $k = 0.001$ and $h = 0.01$. For U_i^M we use the parameters as indicated above.
We see that in the linear case the compact R3C scheme yields better results than the Crank-Nicolson scheme in terms of accuracy, even though the error resulting from the Crank-Nicolson scheme is only slightly bigger (see Table 4.1). Reducing the spatial step size to $h = 0.001$ improves the accuracy considerably, however, it increases the computational time tremendously.

Volatility model	$err_2(Mk)$ with CN	$err_2(Mk)$ with R3C
Constant ($s_i^n = 0$)	0.0016	0.0009
Barles and Soner	0.0006	0.0009
Identity	0.0031	0.0024
Leland	0.0047	0.0056
RAPM	0.0006	0.0005

Table 4.1: ℓ^2 error for different models and schemes.

4.2 American Call option

In this section we want to solve the transformed problem from Section 3.2

$$0 = \Pi_\tau + \big(b(\tau) - \frac{\widetilde{\sigma}^2}{2}\big)\Pi_x - \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x) + r\Pi, \quad x \in \mathbb{R}^+, \quad 0 \le \tau \le T \quad (4.14)$$

with the corresponding volatilities (3.7) subject to the conditions

$$\begin{aligned} &\Pi(x,0) = \begin{cases} -K & \text{for } x < \ln \frac{\varrho(0)}{K} \\ 0 & \text{otherwise} \end{cases} \\ &\Pi(x,\tau) = 0 \quad \text{as } x \to \infty,\ 0 \le \tau \le T, \\ &\Pi(0,\tau) = -K \quad \text{for } 0 \le \tau \le T, \end{aligned} \quad (4.15)$$

and the constraint

$$\varrho(\tau) = \frac{1}{2q}\widetilde{\sigma}^2\Pi_x(0,\tau) + \frac{rK}{q} \quad \text{with} \quad \varrho(0) = \frac{rK}{q}. \quad (4.16)$$

We therefore first describe the solution of (4.14) subject to (4.15) and (4.16) with the corresponding volatilities (3.7) by finite-difference schemes and then present the numerical results.

4.2.1 Finite-Difference Schemes

There have been many approaches to calculate the value of an American option numerically by compact finite-difference schemes in the absence of transaction costs. Recently, Tangman et al. [67, 68] introduced a compact scheme of order $(2,4)$. Two other compact schemes, known as the *Numerov-type* (see [66, 77]) and the *Crandall-Douglas scheme* (see [50]), are analyzed for linear Black-Scholes equations. However, these schemes are not directly transferable to the model in the presence transaction costs. In order to find a solution for the nonlinear Black–Scholes equation (4.14) subject to (4.15) with the corresponding volatilities (3.7) and the constraint (4.16), Ševčovič suggests to combine two approaches that solve the problem for the American Call with a constant volatility numerically [72]. One of them is the transformation of the problem into a variational inequality and its solution by the PSOR algorithm [31, 61]. The other one is the derivation of a nonlinear integral equation for the position of the free boundary without the knowledge of the price itself [47, 76].
Even though these methods are not directly applicable, since they require a constant volatility σ, this approach is successful when it is combined with an operator splitting technique. The idea is to discretize (4.14) in time, to split the equation into a convective and a diffusive part and to find an approximation for the solution pair (Π, ϱ) at each time level. The detailed derivation is given in the sequel.

4.2.1.1 Grid

We discretize the problem (4.14) subject to the conditions (4.15) with the corresponding volatilities (3.7) by confining the unbounded domain $x \in \mathbb{R}^+$ and $\tau \in [0,T]$ to $x \in (0,R)$ with $R > 0$ sufficiently large (see [72]). For the calculation Ševčovič chooses to take $R = 3$, since this is equivalent to $S \in (S_f(t)e^{-R}, S_f(t))$ and yields a good approximation for $S \in (0, S_f(t))$ (as the transformation was $S = S_f(t)e^{-x}$). As previously, we refer to $h > 0$ as the spatial step and to $k > 0$ as the time step, $x_i = ih$, $i \in [0,N]$, $R = Nh$ and $\tau_n = nk$, $n \in [0,M]$, $T = Mk$ (see Figure 4.7).

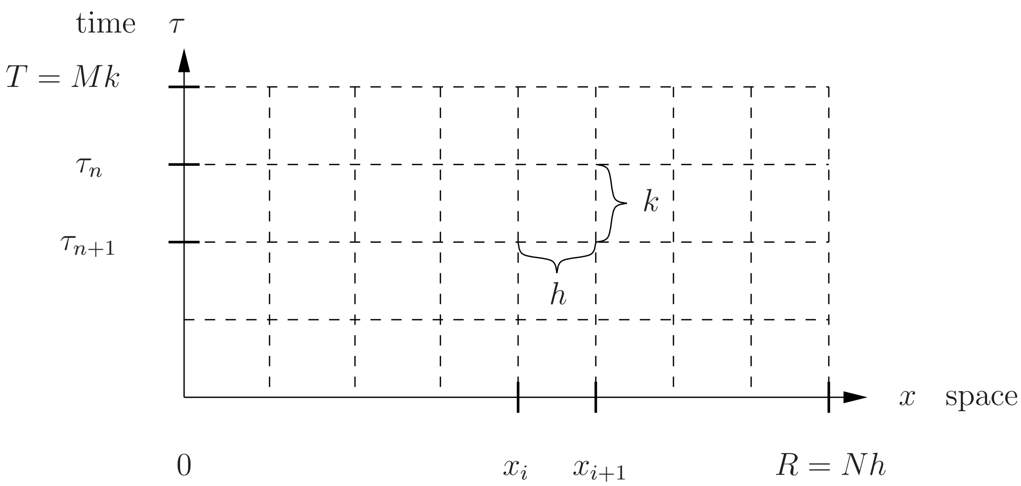


Figure 4.7: Uniform grid for an American Call option.

The approximate solution of (4.14) in x_i at time τ_n is denoted by $\Pi_i^n := \Pi(x_i, \tau_n)$, the value of the free boundary at time τ_n by $\varrho^n := \varrho(\tau_n)$ and the value of the coefficient $b(\tau)$ at τ_n by $b^n := b(\tau_n)$.
We treat the initial and boundary conditions (4.15) in the following way:

$$\begin{aligned} \Pi_i^0 &= \Pi(x_i, 0) = \begin{cases} -K & \text{for } x_i < \ln \frac{\varrho(0)}{K} = \ln \frac{r}{q} \\ 0 & \text{otherwise} \end{cases}, \\ \Pi_0^n &= -K, \\ \Pi_N^n &= 0. \end{aligned} \tag{4.17}$$

4.2.1.2 Difference Quotients

As previously, we denote the forward difference quotient with respect to the spatial variable in x_i at time τ_n with the spatial step size h by:

$$D_h^+ \Pi_i^n := \frac{\Pi_{i+1}^n - \Pi_i^n}{h} \approx \Pi_x(x_i, \tau_n),$$

the backward difference quotient by:

$$D_h^- \Pi_i^n := \frac{\Pi_i^n - \Pi_{i-1}^n}{h} \approx \Pi_x(x_i, \tau_n)$$

and the central difference quotient by

$$D_h^0 \Pi_i^n := \frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} \approx \Pi_x(x_i, \tau_n),$$

omitting the truncation error $\mathcal{O}(h)$, $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively.

4.2.1.3 Volatility Functions

As in the case of the European Call option the volatilities (3.7) can all be written in the form

$$\left(\widetilde{\sigma}_i^n\right)^2 = \sigma^2(1 + s_i^n),$$

where s_i^n denotes the volatility correction in x_i at time τ_n. We choose forward differences to approximate Π_x in the volatility formulae, so that for Leland's model with the volatility (3.7a) our volatility correction becomes

$$s_i^n = Le\,\mathrm{sign}(D_h^+ \Pi_i^n), \tag{4.18a}$$

for the volatility correction in Barles' and Soner's model with the volatility (3.7b) we get

$$s_i^n = \Psi\left(e^{r\tau_n} a^2 D_h^+ \Pi_i^n\right), \tag{4.18b}$$

for the volatility correction in case of treating $\Psi(\cdot)$ as the identity with the original volatility (3.7c) we obtain

$$s_i^n = e^{r\tau_n} a^2 D_h^+ \Pi_i^n, \tag{4.18c}$$

and for the volatility (3.7d) in the Risk Adjusted Pricing Methodology the volatility correction is

$$s_i^n = 3\left(\frac{C^2 M}{2\pi} D_h^+ \Pi_i^n \varrho^n e^{-x_i}\right)^{\frac{1}{3}}. \tag{4.18d}$$

4.2.1.4 Free Boundary

We discretize the free boundary (4.16) by approximating the spatial derivative at the origin $x = 0$ by forward differences and obtain:

$$\varrho^n = \frac{1}{2q}\sigma^2(1 + s_0^n) D_h^+ \Pi_0^n + \frac{rK}{q} \quad \text{with} \quad \varrho^0 = \frac{rK}{q}, \tag{4.19}$$

where s_0^n denotes (4.18) at $x = 0$ depending on the volatility model.
Note, that in case of the RAPM, where the volatility correction is given by equation (4.18d), s_0^n depends on ϱ^n and therefore ϱ^n in (4.19) is expressed by a fixed point equation.

Remark 4.2. *For the American Call option (in contrast to the American Put option) it is possible to derive a series for the location of the optimal* *exercise boundary close to expiry using standard asymptotic analysis (cf. [1, 75]). This local analysis of the free boundary $S_f(t)$ yields*

$$S_f(t) \sim S_f(T)\left(1+\xi_0\sqrt{\frac{1}{2}\sigma^2(T-t)}+\dots\right), \quad \text{as } t \to T, \tag{4.20}$$

*where $\xi_0 = 0.9034\dots$ is a **universal constant** of Call option pricing. Equation (4.20) can be rewritten as*

$$\varrho(\tau) \sim \varrho(0)\left(1+\xi_0\sqrt{\frac{1}{2}\sigma^2(\tau)}+\dots\right), \quad \text{as } \tau \to 0. \tag{4.21}$$

With only very few terms we get a fairly accurate result for the free boundary and thus equation (4.21) will serve us as a check for the case of a constant volatility $\widetilde{\sigma}^2 = \sigma^2$ (see Figure 4.8). Note that this result is especially useful in the first time levels of a numerical calculation where rapid changes in $\varrho(\tau)$ influence the whole solution region.

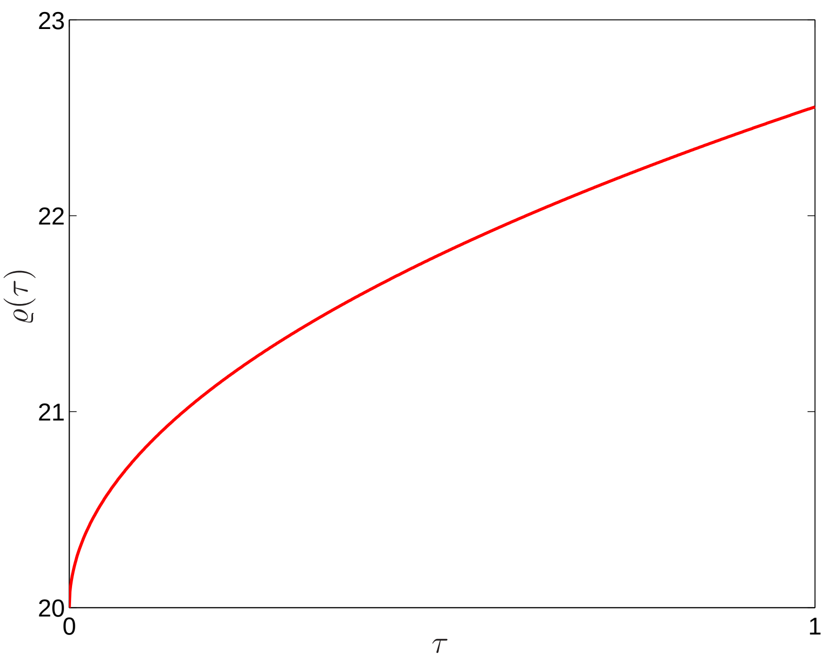


Figure 4.8: Asymptotic solution for the free boundary $\varrho(\tau)$ with $T = 1$, $K = 10$, $\sigma = 0.2$, $r = 0.1$, $q = 0.05$.

4.2.1.5 Splitting in Time Method

We approximate the time derivative of (4.14) by backward differences $D_k^- \Pi_i^n$, the first and second spatial derivatives by central differences $D_h^0 \Pi_i^n$

and $D_h^2\Pi_i^n$. Then, (4.14) becomes:

$$0 = D_k^-\Pi_i^n + \big(b^n - \frac{\sigma^2}{2}(1+s_i^n)\big)D_h^0\Pi_i^n - \frac{1}{2}\partial_x\big(\sigma^2(1+s_i^n)D_h^0\Pi_i^n\big) + r\Pi_i^n \quad (4.22)$$

subject to the Dirichlet conditions (4.17). We introduce an intermediate step at time $\tau_{n-\frac{1}{2}}$, so that

$$D_k^-\Pi_i^n = \frac{\Pi_i^n - \Pi_i^{n-1}}{k} = \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}} + \Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k},$$

and then split the problem (4.22) into a *convective part* with the linear first-order term $b^n D_h^0\Pi_i^n$:

$$0 = \frac{\Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k} + b^n D_h^0\Pi_i^n \quad (4.23)$$

and a *diffusive part* with the nonlinear first- and second-order terms $\sigma^2/2(1+s_i^n)D_h^0\Pi_i^n$ and $-\partial_x\big(\sigma^2/2(1+s_i^n)D_h^0\Pi_i^n\big)$:

$$0 = \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n)D_h^0\Pi_i^n - \frac{1}{2}\partial_x\big(\sigma^2(1+s_i^n)D_h^0\Pi_i^n\big) + r\Pi_i^n. \quad (4.24)$$

Assuming that $D_h^0\Pi_i^n \approx D_h^0\Pi_i^{n-\frac{1}{2}}$, which is reasonable for small time steps k, we can approximate the convective part (4.23) as

$$0 = \frac{\Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k} + b^n D_h^0\Pi_i^{n-\frac{1}{2}}. \quad (4.25)$$

Now the solution to (4.24)-(4.25) gives a good approximation to the solution of (4.22) (see [72]). This decomposition of the problem is called *Lie-Splitting* and is a spitting of order 1 in time.

Convective part:

First, we solve the convective part (4.25), which can be approximated by an explicit solution to the *transport equation*

$$\Pi_\tau + b(\tau)\Pi_x = 0, \quad (4.26)$$

for $(x,\tau) \in \mathbb{R} \times [0,T]$, subject to the boundary and initial conditions

$$\begin{aligned} &\Pi(0,\tau) = -K, \\ &\Pi(x,0) = \begin{cases} -K & \text{for } x < \ln\frac{r}{q} \\ 0 & \text{otherwise} \end{cases} = \Pi^0(x). \end{aligned} \quad (4.27)$$

We then know by the theory of partial differential equations (see e.g. [25]) that the solution for this problem (4.26)–(4.27) is

$$\Pi(x,\tau) = \Pi(x - \int_0^\tau b(s)\,ds, 0) = \Pi^0(x - \int_0^\tau b(s)\,ds) \quad (4.28)$$

with the primitive function $\int b(s)\,ds = B(\tau) + c = \ln \varrho(\tau) + (r-q)\tau + c$. Hence, considering the problem (4.26) for $(x_i, \tau_j) \in \mathbb{R} \times [\tau_{n-1}, \tau_n]$ subject to the boundary and initial conditions

$$\begin{aligned} \Pi(0, \tau_j) &= -K, \\ \Pi(x_i, \tau_{n-1}) &= \Pi^{n-1}(x_i), \end{aligned} \tag{4.29}$$

we know that the solution is given by

$$\begin{aligned} \Pi(x_i, \tau_j) &= \Pi(x_i - \int_{\tau_{n-1}}^{\tau_j} b(s)\,ds, \tau_{n-1}) \\ &= \begin{cases} \Pi(\xi_i^j, \tau_{n-1}) & \text{for } \xi_i^j > 0 \\ -K & \text{otherwise,} \end{cases} \end{aligned} \tag{4.30}$$

where we set $\xi_i^j = x_i - B(\tau_j) + B(\tau_{n-1}) = x_i - \ln \frac{\varrho^j}{\varrho^{n-1}} - (\tau_j - \tau_{n-1})(r-q)$. Then we can write

$$\Pi_i^{n-\frac{1}{2}} = \begin{cases} \Pi(\xi_i^n, \tau_{n-1}) & \xi_i^n = x_i - \ln \frac{\varrho^n}{\varrho^{n-1}} - k(r-q) > 0 \\ -K & \text{otherwise.} \end{cases} \tag{4.31}$$

Here, we use a linear approximation between the discrete values $\Pi(x_i, \tau_{n-1})$, $i \in \mathbb{N}$ in order to compute the value of $\Pi(\xi_i^n, \tau_{n-1})$.
Hence, (4.31) is the solution to the convective part (4.25) of the problem (4.22).

Diffusive part:

We solve the diffusive part (4.24) of the problem (4.22) by the finite-difference method. We approximate the second spatial derivative by central differences $D_h^2\Pi_i^n$ and the first spatial derivative by both central $D_h^0\Pi_i^n$ and backward differences $D_h^-\Pi_i^n$. Then, (4.24) becomes:

$$\begin{aligned} 0 = &\frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} + r\Pi_i^n \\ &- \frac{\sigma^2}{2}\left((1+s_i^n)\frac{\Pi_{i+1}^n - 2\Pi_i^n + \Pi_{i-1}^n}{h^2} + \frac{(1+s_i^n) - (1+s_{i-1}^n)}{h}\frac{\Pi_i^n - \Pi_{i-1}^n}{h}\right) \\ = &\frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} + r\Pi_i^n \\ &- \frac{\sigma^2}{2}\left((1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_i^n}{h^2} - (1+s_{i-1}^n)\frac{\Pi_i^n - \Pi_{i-1}^n}{h^2}\right). \end{aligned}$$

Rearranging leads to a tridiagonal system of equations

$$\Pi_i^{n-\frac{1}{2}} = a_i^n \Pi_{i-1}^n - b_i^n \Pi_i^n + c_i^n \Pi_{i+1}^n, \tag{4.32}$$

with the coefficients

$$\begin{aligned} a_i^n &= \tfrac{\sigma^2}{2}(1+s_i^n)\tfrac{k}{2h} - \tfrac{\sigma^2}{2}(1+s_{i-1}^n)\tfrac{k}{h^2}, \\ b_i^n &= 1 + kr + \tfrac{\sigma^2}{2}(1+s_i^n)\tfrac{k}{h^2} + \tfrac{\sigma^2}{2}(1+s_{i-1}^n)\tfrac{k}{h^2}, \\ c_i^n &= -\tfrac{\sigma^2}{2}(1+s_i^n)\tfrac{k}{2h} - \tfrac{\sigma^2}{2}(1+s_i^n)\tfrac{k}{h^2}. \end{aligned}$$

Equation (4.32) can be written in the form of matrices:

$$\Pi^{n-\frac{1}{2}} = A^n \Pi^n + d^n, \tag{4.33}$$

where

$$\Pi^n = \left(\Pi_1^n, \cdots, \Pi_{N-1}^n\right)^\top \in \mathbb{R}^{N-1},$$

$$A^n = \begin{pmatrix} b_1^n & c_1^n & 0 & \cdots & 0 \\ a_2^n & b_2^n & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{N-2}^n & c_{N-2}^n \\ 0 & \cdots & 0 & a_{N-1}^n & b_{N-1}^n \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)},$$

and

$$d^n = \left(a_1^n \Pi_0^n, \quad 0, \quad \cdots, \quad 0, \quad c_{N-1}^n \Pi_N^n\right)^\top \in \mathbb{R}^{N-1}.$$

Therefore (4.33) solves the diffusive part (4.24) of the problem (4.22).

Now, we have a set of nonlinear equations (4.18), (4.19), (4.31) and (4.33) that delivers the solution to our portfolio $\Pi(x,\tau)$ and to the free boundary $\varrho(\tau)$, from which we can calculate the value of the American Call option $V(S,t)$ with equation (3.8).
In order to see the dependencies of the equations, we rewrite them in the following abstract form:

$$\begin{aligned} s^n &= \mathcal{D}(\Pi^n, \varrho^n), \\ \varrho^n &= \mathcal{F}\big(\Pi^n, s^n\big) = \mathcal{F}\big(\Pi^n, \varrho^n\big), \\ \Pi^{n-\frac{1}{2}} &= \mathcal{G}\big(\Pi^{n-1}, \varrho^n, \varrho^{n-1}\big) = \mathcal{G}\big(\Pi^{n-1}, \varrho^n\big), \\ A\big(s^n\big)\Pi^n = A\big(\Pi^n, \varrho^n\big)\Pi^n &= \Pi^{n-\frac{1}{2}} - d(s^n), \end{aligned} \tag{4.34}$$

where

$$s^n = \left(\ s_0^n, \quad \cdots, \quad s_N^n\ \right)^\top \in \mathbb{R}^{N+1},$$

$\mathcal{D}(\cdot)$ is the right-hand side of (4.18), $\mathcal{F}(\cdot)$ is the right-hand side of (4.19), $\mathcal{G}(\cdot)$ is the right-hand side of the transport equation (4.31), $A(\cdot)$ is the tridiagonal matrix and $d(s^n)$ the vector as defined in (4.33).
As we can see by this notation (4.34), both ϱ^n and Π^n are given in terms of themselves, hence each is given in terms of ϱ^n and Π^n. This problem can be approximately solved by a successive fixed point iteration over $p \in \mathbb{N}$ at each time level n.
Following Ševčovič [72] we define for $n \geq 1$: $\Pi^{n,0} = \Pi^{n-1}$, $\varrho^{n,0} = \varrho^{n-1}$ and $s^{n,0} = s^{n-1}$. Then the (p+1)-th approximation of Π^n, ϱ^n and s^n is obtained as the solution of the system:

$$\begin{aligned} s^{n,p+1} &= \mathcal{D}(\Pi^{n,p}, \varrho^{n,p}), \\ \varrho^{n,p+1} &= \mathcal{F}\big(\Pi^{n,p}, s^{n,p+1}\big), \\ \Pi^{n-\frac{1}{2},p+1} &= \mathcal{G}\big(\Pi^{n-1,p}, \varrho^{n,p+1}\big), \\ A\big(s^{n,p+1}\big)\Pi^{n,p+1} &= \Pi^{n-\frac{1}{2},p+1} - d(s^{n,p+1}). \end{aligned} \tag{4.35}$$

Both the volatility correction $s_i^{n,p+1}$, the free boundary $\varrho^{n,p+1}$ and the solution $\Pi^{n-\frac{1}{2},p+1}$ to the convective part (4.23) can be directly computed from (4.18), (4.19) and (4.31) respectively. The solution $\Pi^{n,p+1}$ to the diffusive part (4.24) has to be calculated from the system of equations (4.33).
Assuming that the system (4.35) converges to some limiting values $s^{n,p_{max}}$, $\varrho^{n,p_{max}}$, $\Pi^{n-\frac{1}{2},p_{max}}$ and $\Pi^{n,p_{max}}$ at each time level n [72], we can calculate $V(S_i,t_n) = V(e^{-x_i}\varrho^n, T-\tau_n)$ with these values and proceed to the next time level $n+1$.
From (3.8) we then know that:

$$V(S_i,t_n) = e^{-x_i}\big(\varrho^n - K + \mathcal{I}_i\big), \tag{4.36}$$

where

$$\begin{aligned}\mathcal{I}_i &= \sum_{j=0}^{i-1}\mathcal{I}_k + \int_{x_{i-1}}^{x_i} e^x\Pi(x,\tau)dx \\ &= \sum_{j=0}^{i-1}\mathcal{I}_k + \frac{x_i - x_{i-1}}{2}\left(e^{x_{i-1}}\Pi_{i-1}^n + e^{x_i}\Pi_i^n\right).\end{aligned}$$

Here, we use the *trapezoidal rule* in order to approximate the integral in equation (3.8).

4.2.1.6 Algorithm

Therefore, we can summarize the calculation of the price $V(S,t)$ for the American Call option in the presence or absence of transaction costs by the following algorithm:

Algorithm 2 Computation of the price $V(S,t)$ for the American Call

Input parameters: R, T, h, k, M, N, r, K, D, σ, Le, a, C, M

solve the ODE (2.13) required for the volatility model of Barles and Soner and interpolate the solution
initialize Π^0
initialize the free boundary $\varrho^0 = rK/q$
transform Π^0 into V^0
set $\pi = \Pi^0$ and $v = V^0$
set $\Pi^{1,0} = \Pi^0$ and $\varrho^{1,0} = \varrho^0$
calculate Π^n, ϱ^n at each time level
for $n = 1 : M$ **do**
    calculate $s^{n,p}$, $\varrho^{n,p}$, $\Pi^{n-1/2,p}$ and $\Pi^{n,p}$ in the successive loop over p
    for $p = 1 : p_{max}$ **do**
        calculate the volatility correction $s^{n,p}$ depending on the volatility model using $\Pi^{n,p-1}$ and $\varrho^{n,p-1}$ (in the case of Barles' and Soner's model use the interpolated solution of (2.13), in the case without transaction costs $s^{n,p} = (0, \cdots, 0)^\top \in \mathbb{R}^{N+1}$)
        calculate $\varrho^{n,p}$ using $\Pi^{n,p-1}$ and $s^{n,p}$
        calculate $\Pi^{n-1/2,p}$ using Π^{n-1} and $\varrho^{n,p}$
        fill the matrix $A^{n,p}$ and the vector $d^{n,p}$ with the corresponding coefficients using $s^{n,p}$
        L-R-decompose $A^{n,p} = L^{n,p} R^{n,p}$
        solve $L^{n,p} y^{n,p} = \Pi^{n-1/2,p} - d^{n,p}$ for $y^{n,p}$
        solve $R^{n,p} \Pi^{n,p} = y^{n,p}$ for $\Pi^{n,p}$
        start over with the loop over p
    end for
    set $\Pi^n = \Pi^{n,p}$ and $\varrho^n = \varrho^{n,p}$
    transform Π^n into V^n
    save the solution in the transformed variables in the array $\pi = \begin{bmatrix} \pi & [-K; \Pi^n; 0] \end{bmatrix}$
    save the solution in the original variables in the array $v = \begin{bmatrix} v & [\varrho^n - K; V^n; 0] \end{bmatrix}$
    start over with the loop over n
end for
plot v at each time level and each stock price, plot ϱ at each time level

4.2.2 Comparison Study

Based on the iterative algorithm described in the previous sections (Algorithm 2), we solve the transformed Black–Scholes equation (4.14) with the corresponding volatilities (3.7) for the American Call option and finally transform $\Pi(x,\tau)$ back to the original option price $V(S,t)$.
The main purpose of this section is to compare the resulting option value $V(S,t)$ and the free boundary $S_f(T-t) = \varrho(\tau)$, that determines the exercise region for the option $V(S,t)$, for the four different transaction cost models (4.18) to the linear model and to each other.
We choose $p_{max} = 5$ for the successive iteration over p in our algorithm in order to solve the system (4.34) with the precision of 10^{-7} [72]. We use the following parameters for our calculations of $\Pi(x,\tau)$ and $\varrho(\tau)$:

$$r = 0.1, \quad \sigma = 0.2, \quad K = 10, \quad T = 1 \text{ (one year)}, \quad R = 3.$$

We start by comparing the free boundary $\varrho(\tau)$ computed with Algorithm 2 to the asymptotic solution (4.21) from Remark 4.2 for the linear case ($s_i^n = 0$). In Figure 4.9 we see that for smaller spatial steps $h \to 0$ the free boundary computed by our iterative algorithm converges monotonically towards the asymptotic solution (4.21) from below.

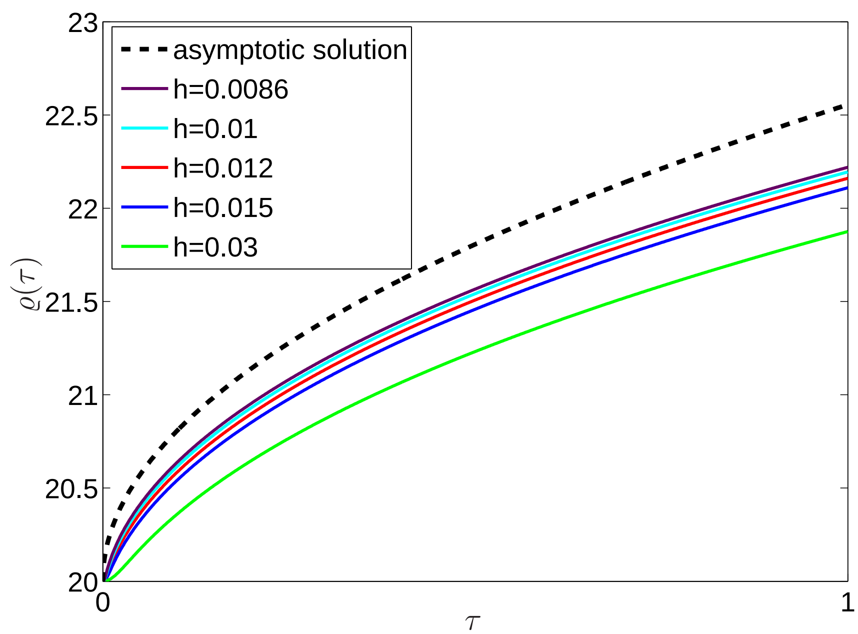


Figure 4.9: Free boundary positions for various spatial steps h with a constant time step $k = 0.0008$ and a constant volatility σ^2 computed by Algorithm 2 vs. the asymptotic solution of (4.21).

We keep the time step $k = 0.0008$ constant and see that for $h = 0.0086$ (purple line) the free boundary at T is computed by our algorithm as

$\varrho(T) \approx 22.2201$. The asymptotic solution at T is $\varrho(T) \approx 22.5552$, which means a relative error of 1.49%. The free boundary values for the other spatial steps can be seen in Table 4.2.

h	0.03	0.015	0.012	0.01	0.0086
$\varrho(T)$	21.8764	22.1111	22.1619	22.1955	22.2201

Table 4.2: Values of the free boundary position for various spatial steps h with a constant time step $k = 0.0008$ and a constant volatility σ^2.

Since the asymptotic solution of (4.21) is only an approximation, we are satisfied by our results and take the free boundary $\varrho(T) \approx 22.1111$ for $k = 0.0008$, $h = 0.015$ (blue line in Figure 4.9) as our reference solution in the absence of transaction costs for the sake of the computational time.
Figure 4.10 shows the structure of the price for the American Call option $V(S,t)$ without transaction costs with $k = 0.0008$ and $h = 0.015$. It is computed with the iterative algorithm described in the previous sections and the parameters above.

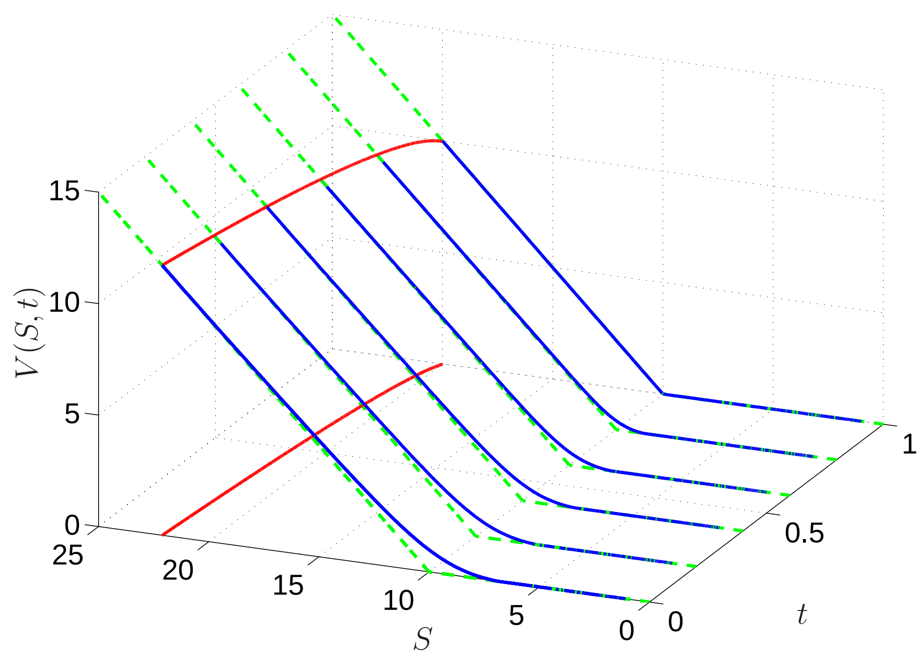
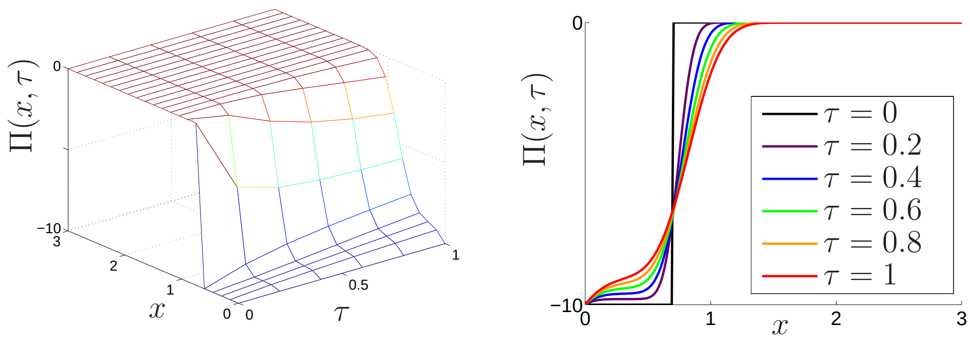


Figure 4.10: Value of an American Call option $V(S,t)$ in the absence transaction costs computed with Algorithm 2 determined by the free boundary (red line).

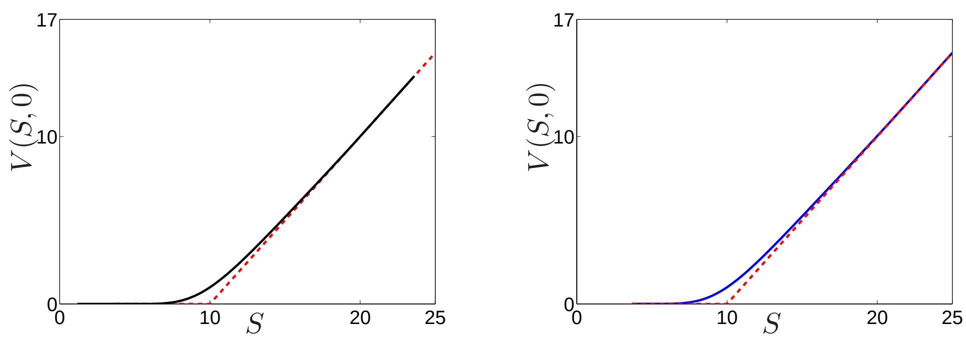
The corresponding synthetic portfolio $\Pi(x,\tau)$ in the absence of transaction costs is illustrated in Figure 4.11. Note, that we include rounding and discretization errors when transforming $\Pi(x,\tau)$ back into $V(S,t)$, since equation (4.36) involves an integral approximation. However, the analysis of $V(S,t)$ is more interesting for us and we therefore assume that these errors are sufficiently small due to the chosen mesh.



(a) 3-D profile. (b) Profile at different time points.

Figure 4.11: Value of the synthetic portfolio $\Pi(x,\tau)$ in the absence of transaction costs computed with Algorithm 2.

We now compare the price $V(S,0)$ computed by Algorithm 2 to the price $V_{PSOR}(S,0)$ computed by the PSOR algorithm in the linear case $s_i^n = 0$. Figure 4.12 shows that with the given mesh size $k = 0.0008$ and $h = 0.015$ the price computed by our algorithm (Figure 4.12(a)) only slightly differs from the price computed by the PSOR algorithm (Figure 4.12(b)).



(a) Computed with Algorithm 2. (b) Computed with PSOR.

Figure 4.12: Price of an American Call option $V(S,0)$ in the absence of transaction costs and the pay-off $V(S,T)$ (red dotted line).

We calculate the error of accuracy of our computation one year to expiry at $t = 0$, denoted by the ℓ^2-error

$$err_2(0) = \left(h \sum_{i=0}^{N} |V_{PSOR}(S_i,0) - V_i^0|^2 \right)^{\frac{1}{2}},$$

where $V_{PSOR}(S_i,0)$ denotes the solution computed by the PSOR algorithm at $S_i = e^{-ih}\varrho(T)$ and $\varrho(T)$ depends on the step size h. For this purpose, we interpolate the solution computed by the PSOR algorithm by the `MATLAB` routines `spline` and `ppval`. For V_i^0 we use our corresponding solution, where $k = 0.0008$. The error can be seen in Table 4.3, which reveals that it is reasonable to assume the accuracy $\mathcal{O}(h)$.

h	0.03	0.015	0.012	0.01	0.0086
ℓ^2-error	0.0365	0.0162	0.0257	0.0084	0.0167

Table 4.3: ℓ^2-error of accuracy of Algorithm 2 compared to the PSOR algorithm in the absence of transaction costs.

We further compute the free boundary profiles for the four different transaction cost models (4.18) by Algorithm 2 and compare them to the profile of the free boundary in the absence transaction costs. For our computations we take $k = 0.0008$ and $h = 0.015$. As expected, we see that for all the transaction cost models the free boundary values are greater than in the case without transaction costs (Figure 4.13). With the given parameters the free boundary in the absence of transaction costs is $\varrho(T) \approx 22.11$, followed by the identity model with $a = 0.02$ ($\varrho(T) \approx 22.16$), Barles' and Soner's model with $a = 0.02$ ($\varrho(T) \approx 22.34$), Leland's model with $\delta t = 0.1$, $\kappa = 0.02$ ($\varrho(T) \approx 22.44$) and finally the RAPM with $C = 0.01$, $R = 30$ ($\varrho(T) \approx 23.39$).

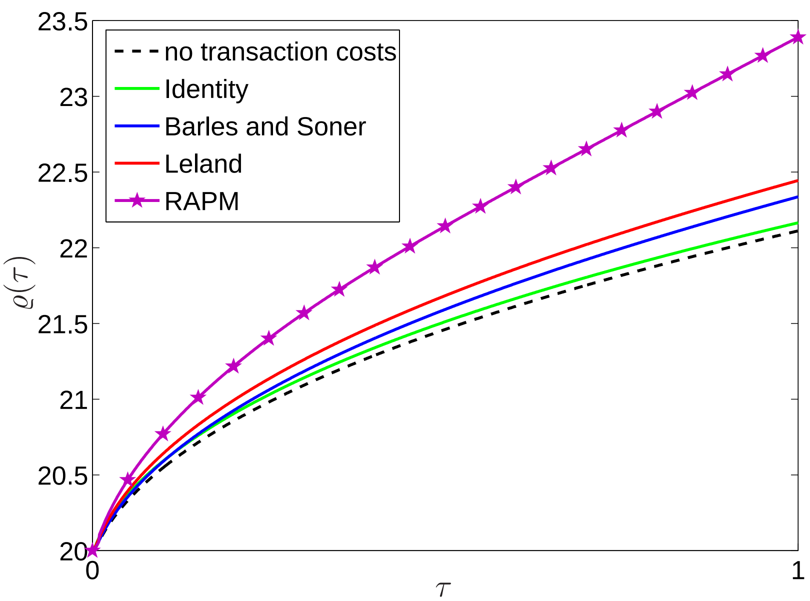


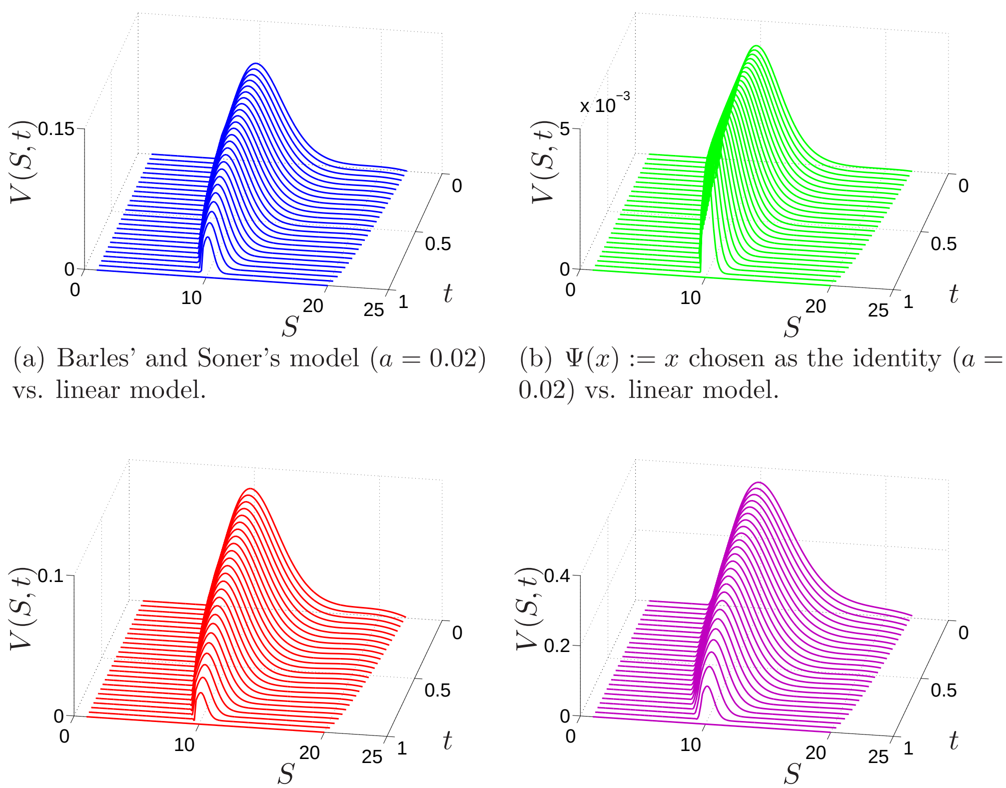
Figure 4.13: Free boundary positions for various transaction cost models vs. the free boundary profile in the absence of transaction costs.

Furthermore, we compute the corresponding values $V(S,t)$ for the American Call option by Algorithm 2 and check the price difference between the

American Call option with transaction costs and the American Call option without transaction costs

$$V_{nonlinear}(S,t) - V_{linear}(S,t).$$

The influence of transaction costs for the four models can be seen in Figure 4.14. We notice that the difference is maximal one year to expiry at $t = 0$ and $S \approx 9.5$. As well as in the case of the European Call option, the difference is not symmetric, but decreases towards the expiry. This seems plausible, since towards expiry the portfolio can not be adjusted as often at it could be adjusted before. Hence, the transaction costs and the value of the American Call option with transaction costs decrease towards $t = 1$.



(a) Barles' and Soner's model ($a = 0.02$) vs. linear model.

(b) $\Psi(x) := x$ chosen as the identity ($a = 0.02$) vs. linear model.

(c) Leland's model ($\delta t = 0.1$, $\kappa = 0.02$) vs. linear model.

(d) RAPM model ($M = 0.01$, $C = 30$) vs. linear model.

Figure 4.14: The influence of transaction costs $V_{nonlinear}(S,t) - V_{linear}(S,t)$.

The corresponding prices $V(S,0)$ in the presence of transaction costs can be seen in Figure 4.15. At $S \approx 9.5$ with the parameters as indicated above and $k = 0.0008$, $h = 0.015$ the price of the American Call option evaluated with the RAPM transaction cost model is the highest (≈ 1.06). It is followed by Barles' and Soner's model (≈ 0.82), Leland's model (≈ 0.78), the identity model (≈ 0.74) and finally the model in the absence of transaction costs (≈ 0.71). As already shown in Table 4.3, the linear price computed by our algorithm (light blue solid line in Figure 4.15) only slightly deviates from the price computed by the PSOR algorithm (black dotted line in Figure 4.15).

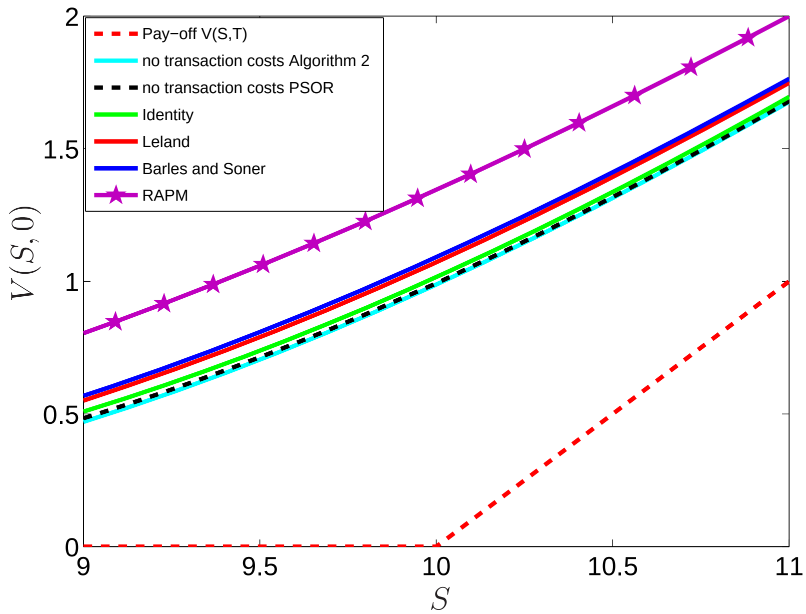


Figure 4.15: Price of an American Call option $V(S,0)$ for different transaction cost models vs. the price without transaction costs.

For other numerical experiments in the future is recommendable to use rather `C` or `C++` in order to reduce the computational time which is relatively high in `MATLAB`.

Summing up, our numerical results show us a considerable price difference between linear and nonlinear prices for both American and European Call options.

Chapter 5

Conclusion

This diploma thesis provided a profound overview over nonlinear Black–Scholes equations for European and American options and the numerical methods for their adequate solution.

We started by introducing the reader to the financial terminology and to Black–Scholes equations in Chapter 1. In Chapter 2 we investigated several reasons for their nonlinearity and focused on the nonlinearity resulting from a modified volatility function due to transaction costs. We concentrated on several transaction cost models, including Leland' model, Barles' and Soner's model, the identity model and the Risk Adjusted Pricing Methodology.

The analytical approach to the solution of nonlinear Black–Scholes equations for the European and American Call option was given in Chapter 3. In order to solve the nonlinear problems numerically we transformed the original problem into a forward convection-diffusion equation with a nonlinear term for the European Call option. The preparation for the numerical solution in case of the American Call option in the presence of transaction costs was achieved by a transformation of the free boundary problem into a fully nonlinear parabolic equation defined on a fixed domain.

In Chapter 4 we introduced the reader to the broad field of numerical approaches to the solution of Black–Scholes equations. We focused on finite-difference schemes.

For the European Call option, we compared several transaction cost models to each other and used two difference schemes for the numerical computation of the option prices. Both the Crank-Nicolson and the R3C scheme provided accurate approximations to the European Call option price. They are unconditionally stable in the linear case and serve as excellent methods for the computation in case of European options in the nonlinear case due to their superiority to standard difference schemes.

For the computation of the prices for American options in a market with transaction costs we used finite difference schemes combined with an operator splitting iterative technique. We compared the influence of transaction costs on the free boundary and the option price for different transaction cost models and obtained substantially higher prices in the presence of

transaction costs.
The obtained results provide a firm basis for further numerical investigations for the solution of nonlinear Black–Scholes equations.

Appendix A

Differential Equations

Since this work deals with several kinds of differential equations, we recall some definitions (see e.g. [25]).

Definition: *An **ordinary differential equation (ODE)** is an equation involving an unknown function of a single variable and its derivatives.*

Definition: *A **stochastic differential equation (SDE)** is a differential equation in which one or more of the terms is a stochastic process (see Appendix B.3).*

Definition: *A **partial differential equation (PDE)** is an equation involving an unknown function of two or more variables and some of its partial derivatives. It is called a k^{th}**-order** PDE if it has the form*

$$F(D^k u(x), D^{k-1} u(x), ..., Du(x), u(x), x) = 0, \tag{A.1}$$

where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is given and

$$u : U \to \mathbb{R}$$

is the unknown. The PDE (A.1) *is called*

*(i) **linear** if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

for given functions a_α ($\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$) and f, $D^k u(x) := \left\{ D^\alpha u(x) || \alpha| = k \right\}$ being the set of all partial derivatives of order k;

(ii) ***semilinear*** *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u, ..., Du, u, x) = 0;$$

(iii) ***quasilinear*** *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, ..., Du, u, x) D^\alpha u(x) + a_0(D^{k-1}u, ..., Du, u, x) = 0;$$

(iv) and ***fully nonlinear*** *if it depends nonlinearly upon the highest order derivatives.*

In this thesis we are mainly interested in second-order PDEs, hence:

Definition: *Let*

$$u(x,t) : D_1 \times D_2 \to \mathbb{R}, \quad D_1, D_2 \subset \mathbb{R},$$

be the unknown function that solves the second-order PDE

$$au_{xx} + 2bu_{xt} + cu_{tt} + du_t + eu_x + fu + g = 0, \tag{A.2}$$

where a, b, c, d, e, f *and* g *are given functions. The PDE* (A.2) *is called*

(i) ***parabolic****, if* $b^2 - ac = 0$*;*

(ii) ***elliptic****, if* $b^2 - ac < 0$*;*

(iii) and ***hyperbolic****, if* $b^2 - ac > 0$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}$ *[31].*

Note, that (A.2) is semi-, quasi- or fully nonlinear depending on the functions a–g.

Appendix B

Stochastics

In this thesis, we use several terms and concepts of probability theory and stochastics. Thus, we recall some definitions (see [31, 61, 62] and the references therein).

B.1 Probability Space

Let Ω be a ***sample space*** representing all possible scenarios (e.g. all possible paths for the stock price over time). A subset of Ω is an *event* and $\omega \in \Omega$ a *sample point*.

Definition: *Let Ω be a nonempty set and $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a σ-**algebra** (not related to the volatility σ), if*

(i) $\Omega \in \mathcal{F}$,

(ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$ and

(iii) whenever a sequence of sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

In our financial scenario, $\mathcal{F}$ represents the space of events that are observable in the market and therefore, all the information available until the time t can be regarded as a σ-algebra $\mathcal{F}_t$. It is logical that $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t < s$, since the information that has been available t is still available at s.

Definition: *Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of Ω. A **probability measure** P is a function that assigns a number in $[0,1]$ to every set $A \in \mathcal{F}$. The number is called the probability of A and is written $P(A)$. We require:*

(i) $P(\Omega) = 1$ and

(ii) whenever a sequence of disjoint sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

The tripel $(\Omega, \mathcal{F}, P)$ is called a ***probability space***.

B.2 Random Variable

Definition: *A real-valued function X on Ω is called a **random variable** if the sets*

$$\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\} = X^{-1}(]-\infty, x])$$

are measurable for all $x \in \mathbb{R}$. That is, $\{X \leq x\} \in \mathcal{F}$.

B.3 Stochastic Process

Definition: *A (continuous) **stochastic process** $X(t) = X(\cdot, t)$, $t \in [0, \infty[$, is a family of random variables $X : \Omega \times [0, \infty[\to \mathbb{R}$ with $t \mapsto X(\omega, t)$ continuous for all $\omega \in \Omega$.*

B.4 Itô Process

Definition: *An **Itô process** is a stochastic process of the form*

$$dX = a(X, t)dt + b(X, t)dW,$$

which is equivalent to

$$X(t) = X(0) + \int_0^t a(X, s)ds + \int_0^t b(X, s)dW,$$

where $X(0)$ is nonrandom, $W(t)$ is a standard Wiener process, $a(\cdot)$ and $b(\cdot)$ are sufficiently regular functions and the integrals are Itô integrals.

B.5 Stopping Time

Definition: *A **stopping time** t is a random variable taking values in $[0, \infty]$ and satisfying*

$$\{t \leq s\} \in \mathcal{F}_s \quad \forall s \geq 0.$$

B.6 Brownian Motion

Definition: *A* ***Brownian motion*** *or* ***Wiener process*** *is a time-continuos stochastic process $W(t)$ with the properties:*

(i) $W(0) = 0$.

(ii) $W(t) \sim \mathcal{N}(0,t)$ *for all* $t \geq 0$. *That is, for each* t *the random variable* $W(t)$ *is normally distributed with mean* $E[W(t)] = 0$ *and variance* $Var[W(t)] = E[W^2(t)] = t$.

(ii) *All increments* $\Delta W(t) := W(t+\Delta t) - W(t)$ *on non-overlapping time intervals are independent. That is,* $W(t_2) - W(t_1)$ *and* $W(t_4) - W(t_3)$ *are independent for all* $0 \leq t_1 < t_2 \leq t_3 < t_4$.

(iv) $W(t)$ *depends continuously on* t.

B.7 Itô's Lemma

Theorem B.1. *Consider a function* $V(S,t) : \mathbb{R} \times [0,\infty[\rightarrow \mathbb{R}$ *with* $V \in C^{2,1}(\mathbb{R} \times [0,\infty[)$ *and suppose that* $S(t)$ *follows the Itô process*

$$dS = a(S,t)dt + b(S,t)dW,$$

where $W(t)$ *is a standard Wiener process. Then* V *follows an Itô process with the same Wiener process* $W(t)$*:*

$$dV = (aV_S + \frac{1}{2}b^2V_{SS} + V_t)dt + bV_S dW, \tag{B.1}$$

where $a := a(S,t)$ *and* $b := b(S,t)$.

If we consider a special case, where $a(S,t) = \mu S$ and $b(S,t) = \sigma S$, then $S(t)$ follows the Geometric Brownian motion, where $W(t)$ is a standard Wiener process, and we have

$$dS = \mu S dt + \sigma S dW.$$

Then, Itô's Lemma yields

$$\begin{aligned} dV &= (\mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + V_t)dt + \sigma S V_S dW \\ &= (\frac{1}{2}\sigma^2 S^2 V_{SS} + V_t)dt + V_S dS. \end{aligned}$$

Appendix C

Pricing Formulae

Theorem C.1. *The solution to the linear Black-Scholes equation* (1.1) *with the terminal and boundary conditions* (1.5)*, or the value of the* ***European Call*** *option, is given by*

$$V(S,t) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{C.1}$$

where

$$d_1 := \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 := \frac{\ln\left(\frac{S}{K}\right) + (r - q - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and $\mathcal{N}(x)$ *is the standard normal cumulative distribution function*

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy, \qquad x \in \mathbb{R}.$$

Respectively, the value of the ***European Put*** *option is the solution to the linear Black-Scholes equation* (1.1) *with the terminal and boundary conditions* (1.6) *and is given by*

$$V(S,t) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2). \tag{C.2}$$

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Berlin, den 28. März 2008

(Unterschrift)