Univ.-Prof. Dr. Matthias Ehrhardt
Dipl.-Math. Alexander Schranner
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Bergische Universität
Wuppertal
Angewandte Mathematik
Numerische Analysis

Introduction to Numerical Methods for Computer Simulation

Exercises No. 2

E 5: *Subordinate matrix norm.*

Given a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, the subordinate matrix norm is defined via

$$\|A\| := \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \qquad (x \in \mathbb{R}^n)$$

for $A \in \mathbb{R}^{n\times n}$.

a) Verify that the subordinate matrix norm is well defined, i.e. the maximum exists.

b) Check the four conditions of a matrix norm:

1. $\|A\| > 0$ for all $A \in \mathbb{R}^{n\times n}$ with $A \neq 0$
2. $\|\alpha A\| = |\alpha| \cdot \|A\|$ for all $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}$
3. $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$
4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$

c) Verify the consistency, i.e.

$$\|Ax\| \leq \|A\| \cdot \|x\| \quad \text{for all } \; A \in \mathbb{R}^{n\times n}, \; x \in \mathbb{R}^n.$$

E 6: *Special matrices.*

a) The matrix $A \in \mathbb{R}^{n\times n}$ is called symmetric, if $A^T = A$ holds. Prove that symmetric matrices A satisfy

$$\|A\|_2 = \rho(A),$$

where $\|\cdot\|_2$ is the subordinate matrix norm of the Euclidean vector norm and $\rho(A)$ the spectral radius of A. In addition, verify that, assuming $\det A \neq 0$, if follows

$$\kappa_2(A) = \frac{\max\{|\lambda_1|, \ldots, |\lambda_n|\}}{\min\{|\lambda_1|, \ldots, |\lambda_n|\}}$$

with the condition number κ_2 corresponding to the Euclidean norm and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.

b) The matrix $Q \in \mathbb{R}^{n \times n}$ is called orthogonal, if $Q^{-1} = Q^T$ holds. Verify the properties

$$\|Q\|_2 = 1 \quad \text{and} \quad \kappa_2(Q) = 1.$$

c) Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Prove that A^{-1} is symmetric and positive definite, too.

E 7: *Elementary matrix operations.*

Consider the class of real elementary matrices, i.e. scaling matrices D, transposition matrices P_{ij} and row/column operators $N_{ij}(\alpha)$.

a) Specify the determinants of these matrices.

b) Show that each lower triangular matrix L satisfying $\det L \neq 0$ can be written as the product of elementary matrices.
Remark: An arbitrary matrix A with $\det A \neq 0$ can be written as the product of elementrary matrices, too, which is harder to prove.

E 8: *Transformed norms.*

Let $\|x\|$ be a vector norm on $\mathbb{R}^n$ and $\|A\|$ the subordinate matrix norm on $\mathbb{R}^{n \times n}$.

a) Show that, given a matrix $M \in \mathbb{R}^{n \times n}$ with $\det(M) \neq 0$, the function

$$\|\cdot\|_M : \mathbb{R}^n \to \mathbb{R}, \quad \|x\|_M := \|Mx\|$$

represents a vector norm.

b) Write the subordinate matrix norm $\|A\|_M$ corresponding to $\|x\|_M$ in a form using the original matrix norm $\|\cdot\|$ and the matrices A, M.

c) Given the matrices

$$A = \begin{pmatrix} 0.9 & 100 \\ 0 & 0.9 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \quad \text{with } \delta > 0,$$

choose a vector norm and δ such that $\|A\|_D = 0.91$ is satisfied in the corresponding transformed matrix norm.