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Bergische Universität
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Angewandte Mathematik
Numerische Analysis

Introduction to Numerical Methods for Computer Simulation

Exercises No. 7

E 19: *Ordinary Gradient Method.*

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

The global minimum of the function

$$f(x) := \tfrac{1}{2} x^T A x - x^T b$$

is just $x^* = A^{-1}b$, since $\operatorname{grad} f(x) = Ax - b$ holds. An approximation of the minimum point can be determined iteratively using the ordinary gradient method:

$$\begin{aligned} &k = 0, 1, 2, \ldots \\ r^k &= b - Ax^k \\ \sigma^k &= \frac{{r^k}^T r^k}{{r^k}^T A r^k} \\ x^{k+1} &= x^k + \sigma^k r^k \end{aligned}$$

a) Perform two steps of the ordinary gradient method using $x^0 = (1, 1)^T$.

b) Prove that the directions r^0 and r^1 are orthogonal.

c) How many matrix-vector-multiplications, scalar products, scalar-vector-multiplications and vector-vector-additions/subtractions are required to execute l steps?

d) Sketch the iteration steps in a coordinate system including the contour lines of the function $f(x)$, which run through the points x^0, x^1, x^2.

E 20: *Conjugate Gradient Method.*

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

The conjugate gradient (CG) method to solve the linear system iteratively reads:

$$\begin{aligned}
&p^0 = r^0 = b - Ax^0 \\
&k = 0, 1, 2, \dots \\
&\qquad \alpha^k = \frac{{r^k}^T r^k}{{p^k}^T A p^k} \\
&\qquad x^{k+1} = x^k + \alpha^k p^k \\
&\qquad r^{k+1} = r^k - \alpha^k A p^k \\
&\qquad \beta^k = \frac{{r^{k+1}}^T r^{k+1}}{{r^k}^T r^k} \\
&\qquad p^{k+1} = r^{k+1} + \beta^k p^k
\end{aligned}$$

a) Perform two steps of the conjugate gradient method using $x^0 = (1, 1)^T$.

b) Prove that the directions p^0 and p^1 are A-orthogonal.

c) How many matrix-vector-multiplications, scalar products, scalar-vector-multiplications and vector-vector-additions/subtractions are required to execute l steps?

d) Sketch the iteration steps in a coordinate system including the contour lines of the function $f(x) = \frac{1}{2}x^T Ax - x^T b$, which run through the points x^0, x^1, x^2.

E 21: *Krylov Spaces.*

Let $A \in \mathbb{R}^{n\times n}$ be a regular matrix ($\det A \neq 0$). Given an arbitrary $y \in \mathbb{R}^n$, the corresponding jth *Krylov space* is defined by

$$\mathcal{K}^j(y, A) := \text{span}\{y, Ay, A^2y, \dots, A^{j-1}y\} \quad \text{for} \quad j = 1, 2, \dots.$$

According to the linear system $Ax = b$ with $x, b \in \mathbb{R}^n$ and an arbitrary $x^0 \in \mathbb{R}^n$, let $r^0 := b - Ax^0$ be the residual.

Prove that

$$\mathcal{K}_j(r^0, A) = \mathcal{K}_{j+1}(r^0, A)$$

implies $x^* := A^{-1}b \in x^0 + \mathcal{K}_j$, i.e. $x^* = x^0 + v$ with $v \in \mathcal{K}_j$ holds.