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Bergische Universität
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Numerische Analysis

Introduction to Numerical Methods for Computer Simulation

Exercises No. 8

E 22: *Computation of square roots via Newton's method.*

The square root of a positive real value a is given by the zero of the nonlinear equation $f(x) \equiv x^2 - a = 0$.

a) Use the Newton method $x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$ to solve the problem involving $a = 2,\ x^0 = 1$ and compute the first four steps.

b) Use the simplified Newton method $x^{k+1} = x^k - \frac{f(x^k)}{f'(x^0)}$ to solve the problem involving $a = 2,\ x^0 = 1$ and compute the first four steps.

Compare the results of both methods to the exact solution.
What are the reasons for the arising behaviour?

E 23: *Convergence of Newton's method.*

The Newton iteration applied to a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^{m+1}(\mathbb{R})$ reads

$$x^{k+1} = \Phi(x^k) := x^k - \frac{f(x^k)}{f'(x^k)}, \quad k \in \mathbb{N}_0. \qquad (\star)$$

a) Consider the two functions

$$f_1(x) = (x-1)^2 = x^2 - 2x + 1 \quad \text{and} \quad f_2(x) = (x-1)(x+1) = x^2 - 1.$$

Perform three steps of Newton's method with starting value $x^0 = 2$.
For which function does the technique converge faster to the exact zero $\hat{x} = 1$?

b) Let $\hat{x}$ be a simple zero of f (i.e. $f(\hat{x}) = 0; f'(\hat{x}) \neq 0$).
Determine the order p of local convergence in the Newton method $(\star)$.

c) Let $\hat{x}$ be a multiple zero of f with multiplicity m
(i.e. $f^{(j)}(\hat{x}) = 0$ for $j = 0, 1, \ldots, m-1$; $f^{(m)}(\hat{x}) \neq 0$).
Determine the order p of local convergence with respect to m. Specify a local statement about the dominating coefficient (corresponding to factor $(x^k - \hat{x})^p$).

d) In a special variant of Newton's method

$$x^{k+1} = \Phi(x^k) := x^k - \gamma \frac{f(x^k)}{f'(x^k)}, \quad k \in \mathbb{N}_0, \qquad (\star\star)$$

the ordinary correction is scaled by the constant $\gamma \in \mathbb{N}$.
For which m of part c) does $(\star\star)$ converge? Determine the order of local convergence and, in case of linear convergence, the dominating coefficient in dependence on m, γ.

E 24: *Accuracy of Newton's method.*

The zeros of $f(x) \equiv \frac{1}{x} - a$ $(a \in \mathbb{R})$ shall be calculated by Newton's method.

a) Show that the iteration formula reads

$$x^{k+1} = x^k + x^k(1 - ax^k), \quad k \in \mathbb{N}_0.$$

b) Show that the error $e^k := x^k - \frac{1}{a}$ satisfies the recursion

$$e^{k+1} = -a(e^k)^2, \quad k \in \mathbb{N}_0.$$

Employing induction, prove the statement

$$e^k = -\frac{1}{a}\rho^{2^k}, \quad k \in \mathbb{N} \quad \text{with} \quad \rho := |ax^0 - 1|.$$

Which condition for ρ and x^0 is equivalent to the global convergence of the iteration?

c) Let $\frac{1}{2} \leq a \leq 1$ and $x^0 = 1.5$.
Determine the maximum number of required iteration steps, which are necessary to obtain an approximation x^k of $\frac{1}{a}$ with 24 or 56 correct dual digits, repectively.
What is the corresponding number of required floating point operations?