

Exercise Sheet 5 to the Lecture Course “Computational Finance”
(Monte Carlo Simulations)

Task 1 (Implementing Euler’s Method) (4 Points)

Implement the method of Euler & Maruyama to discretize an SDE. Consider the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with $S_0 = 50$, $\mu = 0.1$ and $\sigma = 0.2$. Compute 10 paths for $t_0 = 0$ and a time increment $\Delta t = 0.01$ for $0 \leq t \leq 1$. and plot the calculated trajectories.

Task 2 (3 Points)

By transformation of two independent standard normally distributed random variables $Z_i \sim \mathcal{N}(0, 1)$, $i = 1, 2$, two new random variables are obtained by

$$\Delta \widehat{W} := Z_1 \sqrt{\Delta t}, \quad \Delta \widehat{Y} := \frac{1}{2}(\Delta t)^{3/2} \left(Z_1 + \frac{1}{\sqrt{3}} Z_2 \right).$$

Show that $\Delta \widehat{W}$ and $\Delta \widehat{Y}$ have the moments

$$E(\Delta Y) = 0, \quad E(\Delta Y^2) = \frac{1}{3}(\Delta t)^3, \quad E(\Delta Y \Delta W) = \frac{1}{2}(\Delta t)^2.$$

Task 3 (Error of the Milstein Scheme) (3 Points)

To which formula does the Milstein scheme reduce for linear SDEs?

Use the Milstein scheme to perform the experiment outlined in Example 3.2 from the lecture course, i.e. use the parameters $X_0 = 50$, $\alpha = 0.06$, $\beta = 0.3$, $T = 1$ for $N = 50$ simulations for the linear SDE

$$dX_t = \alpha X_t dt + \beta X_t dW_t, \quad \text{initial value } X_0 \quad \text{for } t = 0.$$

Compare the solution with the analytical solution

$$X_t = X_0 \exp\left(\left(\alpha - \frac{1}{2}\beta^2 t + \beta W_t\right)\right). \quad (3.2)$$

fed with the same Wiener process to determine the estimate $\hat{\varepsilon}(h)$.

Set up a table similar as this table:

Table of the $\hat{\varepsilon}(h)$	$h=0.01$	$h=0.005$	$h=0.002$	$h=0.001$	$h=0.0005$
series 1 (with seed ₁)
series 2 (with seed ₂)
series 3 (with seed ₃)

to show

$$\varepsilon(h) \approx h$$

holds for Example 3.2.

- **Return** the solutions until Monday, December 5, **before** the lectures.