Prof. Dr. M. Ehrhardt
D. Shcherbakov, M.Sc.
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Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation II: Partial Differential Equations (PDEs)

Exercise Sheet 4- Tridiagonal matrices, discrete maximum principle

Return of Exercise Sheet: May 24, 2012 (before the lecture)

Homework 10: *Tridiagonal matrices* (2 Points)

Show that the eigenvalues and the corresponding eigenvectors of the *tridiagonal matrix*

$$T = \mathrm{tridiag}(a, b, c) = \begin{pmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{pmatrix}_{N\times N}$$

are given by

$$\lambda_j = b + 2c\sqrt{\frac{a}{c}}\cos\frac{j\pi}{N+1}$$

and $u_j = (u_1, \dots, u_k, \dots, u_N)^\top$ with

$$u_k = 2\left(\sqrt{\frac{a}{c}}\right)^k \sin\frac{kj\pi}{N+1}, \qquad k = 1, \dots, N, \quad j = 1, \dots, N.$$

Homework 11: (3 Points)

Discretize the homogeneous heat equation with homogeneous Dirichlet boundary

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 \\ u(x, 0) = u_0(x), & 0 < x < 1 \\ u(0, t) = 0, & t > 0 \\ u(1, t) = 0, & t > 0 \end{cases}$$

on an arbitrary spatial grid

$$0 = x_0 < x_1 < \dots < x_J = 1$$

and a uniform temporal grid using the implicit Euler method and the approximation

$$u''(x) \approx \frac{2}{h_j + h_{j+1}}\left(\frac{u_{j+1} - u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j}\right).$$

Write down the linear system of equations for u_j^{n+1} and show that the coefficients matrix is irreducible diagonal dominant and the real part of the eigenvalues is stricly larger than zero.

Homework 12: *ℓ^2–Stabilität* (5 Points)

We consider a finite difference scheme for the *convection–diffusion equation*

$$\begin{aligned} u_t &= a u_{xx} - b u_x, \qquad 0 < x < 1, \quad t > 0 \\ u(0,t) &= u(1,t) = 0, \end{aligned}$$

where $a > 0$. For simplicity we assume $b > 0$.
Rewrite the scheme $D_t^+ u_j^n = a D_x^2 u_j^n - b D_x^0 u_j^n$, $j = 1, \ldots, J-1$, in the form

$$u_j^{n+1} = (1 - 2a\gamma) u_j^n + a\gamma(1 - P_e) u_{j+1}^n + a\gamma(1 + P_e) u_{j-1}^n, \qquad P_e = \frac{bh}{2a}$$

and prove that the scheme has the following properties

$$P_e \leq 1 \implies \max_{0 \leq j \leq J} |u_j^{n+1}| \leq \max_{0 \leq j \leq J} |u_j^n|$$

(in case that the ℓ^2–stability condition $a\gamma \leq 1/2$ for $\gamma := k/h^2$ is satisfied).
State an example to illustrate that the above maximum norm estimate does not hold in general for $P_e > 1$.