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Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation II: Partial Differential Equations (PDEs)

Exercise Sheet 6- Discretizations of higher Order

Return of Exercise Sheet: June 14, 2012 (before the lecture)

Homework 16: (6 Points)

Consider the following second order boundary value problem

$$-u'' = f \quad \text{in} \quad \Omega = (0,1)$$
$$u(0) = u(1) = 0$$

and any 3 point approximation

$$\frac{1}{h^2}\big(\alpha_{i,0}u_{i-1} + \alpha_{i,1}u_i + \alpha_{i,2}u_{i+1}\big) = \sum_{j=1}^{J} \beta_{i,j} f(\tau_{i,j})$$
$$u_0 = u_N = 0$$

on the interval $[x_{i-1}, x_{i+1}]$, where $\alpha_{i,0}, \alpha_{i,1}, \alpha_{i,2}, \beta_{i,j}$ are parameters to be determined and $\tau_{i,j} \in [x_{i-1}, x_{i+1}]$ are auxiliary points. For the scheme we require:

1. The scheme is exact on a L+1 dimensional space S, i.e.

$$\frac{1}{h^2}\big(\alpha_{i,0}s_\ell(x_{i-1}) + \alpha_{i,1}s_\ell(x_i) + \alpha_{i,2}s_\ell(x_{i+1})\big) = -\sum_{j=1}^{J} \beta_{i,j} s''_\ell(\tau_{i,j})$$

   for $\ell = 0, \dots, L$, where $s_0, s_1, \dots, s_L$ denotes a basis of S.
   Usually, S is a subspace of the space of polynomials:

   Lemma: *If S is the space of polynomials of degree $\leq L$, then the order of consistency of the method is $L-1$.*

2. Normalization condition $\sum_{j=1}^{J} \beta_{i,j} = 1$.

Choose a polynomial basis and solve the following tasks:

1. Which method yields $J = 1$, $\tau_{i,1} = x_i$?
2. How to choose $\tau_{i,j}$ such that with $J = 3$ one obtains the well-known *method of fourth order*

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \frac{1}{12}\big(f(x_{i-1}) + 10f(x_i) + f(x_{i+1})\big)$$

3. Is it possible to obtain with $J = 3$ a higher order of consistency?

Hint: Assume without loss of generality $x_i = 0$ and consider the interval $[-h, h]$. Construct a basis of the space of polynomials having zeros at the 'right' locations.

Homework 17: (4 Points)

Rewrite the *scheme of higher order* from the lecture course:

$$\begin{aligned} \Lambda' u(x) &= -\varphi(x), \qquad && x \in \Omega_h \\ u(x) &= \mu(x), && x \in \Gamma_h \end{aligned}$$

with

$$\begin{aligned} \Lambda' &= \Lambda_1 + \Lambda_2 + \frac{h_1^2 + h_2^2}{12} \Lambda_1 \Lambda_2 \\ \varphi &= f + \frac{h_1^2}{12} \Lambda_1 f + \frac{h_2^2}{12} \Lambda_2 f \end{aligned}$$

in the form of the *discrete maximum principle.* Analyse under which conditions for h_1, h_2 are the assumptions of the maximum prinziples fulfilled.