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Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation II: Partial Differential Equations (PDEs)

Exercise Sheet 8- Friedrichs Inequality, Poincare and Sobolev Inequality

Return of Exercise Sheet: June 28, 2012 (before the lecture)

Homework 21: (4 Points)

1. Prove the following *inequality of Wirtinger*: Let $u \in C^1([0, 2\pi])$, $u(0) = u(2\pi)$ und $\int_0^{2\pi} u(t)\,dt = 0$. Then
$$\int_0^{2\pi} u(t)^2\,dt \le \int_0^{2\pi} u'(t)^2\,dt.$$
Hint: Use for u a Fourier expansion
$$u(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt$$
and the Parseval equation.

2. Further show that from this inequality for $f \in C^1([a, b])$, assuming that $f(a) = f(b) = 0$, the *classical Friedrichs inequality*
$$\int_a^b f(x)^2\,dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b f'(x)^2\,dx.$$
follows.

Homework 22: (4 Points)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Show that:

1. A norm on $V = H_0^1(\Omega)$ is defined by
$$\|u\|_{\Omega,c} = \left(\int_\Omega |\nabla u|^2 + c(x)\,u^2\,dx\right)^{\frac{1}{2}},$$
if $c \in L^\infty(\Omega)$ and $c \ge 0$ almost everywhere.

2. The Laplace operator is *coercive* on $(V, \|u\|_{\Omega,c})$, i.e. it exists a constant $c_2 > 0$ with
$$\int_\Omega |\nabla u|^2\,dx \ge c_2\,\|u\|_{\Omega,c}^2.$$

3. How does the coercivity constant depend on the used norm?

Homework 23: (3 Points)

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x = r\cos\varphi,\ y = r\sin\varphi,\ r \in (0, 1),\ \varphi \in (0, \omega)\}$.

The function $u(x, y) = r^{\frac{\pi}{\omega}} \sin(\frac{\pi}{\omega}\varphi)$ is the classical solution of the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad x \in \Omega \\ u &= 0 \quad x \in \Gamma_1 \cup \Gamma_2 \\ u &= \sin(\frac{\pi}{\omega}\varphi) \quad x \in \Gamma_3. \end{aligned}$$

1. For which ω is $u \in H^2(\Omega)$?
2. Does Ω have a Lipschitz boundary?