Technical report, IDE0915 , September 17, 2009

A high-order compact method for nonlinear Black-Scholes option pricing equations with transaction costs

Master’s Thesis in Financial Mathematics

Ekaterina Dremkova

MASTER'S PROGRAMME IN FINANCIAL MATHEMATICS

School of Information Science, Computer and Electrical Engineering
Halmstad University

A high-order compact method for nonlinear Black-Scholes option pricing equations with transaction costs

Ekaterina Dremkova

Halmstad University

Project Report IDE0915

Master's thesis in Financial Mathematics, 15 ECTS credits

Supervisor: Priv.-Doz. Dr. Matthias Ehrhardt
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Daniel Ševčovič

September 17, 2009

Department of Mathematics, Physics and Electrical engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University

Contents

1 Introduction **1**

2 Compact schemes **3**
2.1 The valuation of options 3
2.2 Transaction cost models 4
2.3 High-order compact schemes 5
2.3.1 Classical finite difference schemes 7
2.3.2 Compact schemes of higher order 10
2.4 The fixed domain transformation 14
2.5 The method of Liao and Khaliq 17

3 Numerical solution **21**
3.1 American options 22
3.2 Grid 22
3.3 Free boundary 23
3.4 Method of Liao and Khaliq for American Options 24
3.5 Algorithm 25

4 Conclusions **27**

Notation **29**

Bibliography **31**

Chapter 1

Introduction

Over the last several years there were numerous discussions about the Black-Scholes model and its appliance to the nonlinear case. In the linear model there exists a lot of restrictions such as frictionless, liquid and complete market. But all these assumptions are not fulfilled in reality. One of the most important issues that has a great influence on the option pricing strategy are transaction costs.

Several authors proposed relaxing hedging conditions dealing with transaction costs (see [1]). The first one was Leland [18]: he showed that

$$\frac{\kappa}{2}|\delta\Delta|S = \frac{\sigma^2}{2} Le S^2 V_{SS} \delta t. \tag{1.1}$$

Here, Le denotes the *Leland number*:

$$Le = \sqrt{\frac{2}{\pi}} \left(\frac{\kappa}{\sigma\sqrt{\delta t}} \right). \tag{1.2}$$

Initiating this number and inserting it into the nonlinear Black-Scholes equation, Leland deduces that the option price is the solution of:

$$0 = V_t + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS} + rSV_S - rV, \tag{1.3}$$

where $\tilde{\sigma}^2$ is the *modified volatility*

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + Le \ signV_{SS}\right). \tag{1.4}$$

Although this model has been criticized by Kabanov [15], it has played a significant role in financial mathematics.

Boyle and Vorst [5] proposed the following form of modified volatility:

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + Le\sqrt{\frac{\pi}{2}} sign(V_{SS})\right). \tag{1.5}$$

Like Leland, they considered that V is convex and $\tilde{\sigma}^2 = \sigma^2(1 + Le\sqrt{\pi/2})$. Under these assumptions the nonlinear Black-Scholes equation reduces to a linear equation.

Hodges and Neuberger used an *utility function* [1]. They don't define it, but assume that this function characterizes the behavior of the investor. In other words, it is a measurement of investor sufficing from the input. They assume a market with transaction costs and state that the option price should be equal to the unique cash increment. This is a displacement from the difference between the highest utility from the final wealth with and without option liability.

Barles and Soner [3] present another model by using the utility function approach of Hodges and Neuberger. They make the following assumptions: $\kappa = a\sqrt{\epsilon}\ \ \forall a > 0$, as $\epsilon \to 0, \kappa \to 0$, V is the unique viscosity solution of the nonlinear Black-Scholes equation (1.3) with

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + \Psi(e^{r(T-t)}a^2S^2V_{SS})\right). \tag{1.6}$$

Here, $\Psi(x)$ is the solution of the nonlinear ordinary differential equation (ODE)

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x)} - x}, \quad x \neq 0, \Psi(0) = 0. \tag{1.7}$$

There is another model proposed by Kratka in [17] and improved by Jandačka and Ševčovič in [14], which is called the Risk Adjusted Pricing Methodology (RAPM). Here we need to consider transactions and find the optimal time-lag δt between them. Then we sum up the rates of the transaction costs and of the risk from an unprotected portfolio. The aim is to minimize this sum. In this way the portfolio is well protected with RAPM and we have a new form of the modified volatility

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + 3(\frac{C^2M}{2\pi}SV_{SS})^{\frac{1}{3}}\right), \tag{1.8}$$

where $M \geq 0$ is the measurement of the transaction cost and $C \geq 0$ is the measurement of the risk premium.

Further we will consider compact schemes for European and American options and work closer with the transaction cost model of Barles and Soner. Instead of solving the singular differential equation (1.7) we propose to use some properties of $\Psi = \Psi(A)$ described recently in [6]. As compact schemes cannot be directly applied to American type options and multi-dimensional problems, we will try to employ them using a fixed domain transformation explained in [2]. We will also consider the method of Liao and Khaliq [19] for solving nonlinear Black-Scholes equation with transaction costs.

Chapter 2

Compact schemes

2.1 The valuation of options

Trading of options on assets first was organized in 1973. Assets that lie in base of these financial derivatives are stocks, stock indices, foreign currency, debt instruments, goods and future contracts. There exists two types of vanilla options. *Call option* is a contract that gives the right to the holder to buy the underlying asset on a particular date at a particular value. *Put option* is a contract that gives the right to the holder to sell the underlying asset on a particular date at a particular value. The price in the contract is called *exercise price* (or *strike price*). The date in the contract is called *expiration date* (or *exercise date, maturity date*). Options can also be two types according to the expiration date. *American options* may be exercised at any time before maturity date, *European options* may only be exercised at the maturity date.

Remark 1. *The terms "American" and "European" do not mean the place of contracting but solely the type of the option.*

Most options traded are American type options. Whereas European option are easier to analyze and some of the properties of American type options are carried over from the properties of European options.

Remark 2. *It should be said that options give the right to the holder to do something. But it does not mean that the holder has to do it. This fact separates options from futures and forwards, where the holder has to buy or sell the underlying asset.*

Let us mention also that to buy the option contract, the investor has to pay. There exists two sides in every option contract. From one side there is an investor who has *the long position*, i.e. he has bought the option or he is the *holder*. From the other side there is an investor who has the short position, i.e. he has sold or he is the *writer*. The holder of an option receives the money immediately, but has the potential liabilities later. His gains or losses are controversal to those of the writer.

Often it is beneficial to characterize the positions of European option through the profit of the holder. In this case the initial value of the option is not considered in calculations. If K is the exercise price, S_T - the final value of the underlying asset, then in European call option the long position will have the profit $\max(S_T - K, 0)$. This shows that the option will be exercised if $S_T > K$, and will not be exercised if $S_T \leq K$. The writer of the option will have the opposite profit, i.e. in this case $-\max(S_T - K, 0) = \min(K - S_T, 0)$. The profit of the long position in the European put option is $\max(K - S_T, 0)$.

2.2 Transaction cost models

When Fischer Black and Myron Scholes introduced their famous equation in [4] it became easier to evaluate options. In an ideal market the price of an option can be obtained through the *linear Black-Scholes equation*

$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \tag{2.1}$$

where $S := S(t) > 0$ and $t \in (0, T)$. This equation also gives a hedging portfolio that replicates the contingent claim.

But in the real world there are a lot of restrictions such as frictionless, liquid and complete market, transaction costs, and both volatility σ and the drift μ can depend on the stock price S, the time t or the derivatives of the option price V itself.

Here we will consider the famous Barles-Soner transaction cost model for the nonlinear case of the Black-Scholes model with a constant drift μ and a *modified volatility function* proposed in [3]

$$\sigma = \sigma_0(1 + \Psi[\exp(r(\tau_0 - \tau))a^2 S^2 V_{SS}]), \tag{2.2}$$

where r is a risk-free interest rate, τ_0 the maturity, $a = \mu\sqrt{\gamma N}$, γ risk averse factor, N number of options to be sold. Under these assumptions (2.1) becomes the following *nonlinear Black-Scholes equation*

$$0 = V_t + \frac{1}{2}\sigma^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV, \tag{2.3}$$

where $dS = \mu S dt + \tilde{\sigma} S dW,\ S > 0,\ t \in (0, T)$.

2.3 High-order compact schemes

In [9] Düring, Fournié and Jüngel presented an approach to use high-order compact schemes which need a stencil of three points in space only. Consider the equation

$$0 = V_t + \frac{1}{2}\sigma(V_{SS})^2 S^2 V_{SS} + rSV_S - rV, \tag{2.4}$$

where the nonlinear volatility $\sigma(V_{SS})$ is given by (2.2). This equation is backward in time because it is solved for the price $S \geq 0$ of the underlying asset and time $0 \leq t \leq T$. The terminal and boundary conditions are the following:

$$\begin{aligned} V(S,T) &= V_0(S),\ S \geq 0, \\ V(0,t) &= 0,\ 0 \leq t \leq T, \\ V(S,t) &\sim S - Ee^{r(t-T)} \ \ (S \to \infty). \end{aligned} \tag{2.5}$$

The last condition can be rewritten in the form

$$\lim_{S\to\infty} \frac{V(S,t)}{S - Ee^{r(t-T)}} = 1$$

uniformly for $0 \leq t \leq T$.

Let us look at the problem in the nonlinear case and consider the Barles-Soner nonlinear volatility (2.2). In the Black-Scholes equation we apply a time reversal $\tau = T - t$ and obtain the following nonlinear model

$$0 = U_\tau - \frac{1}{2}\sigma^2 S^2 U_{SS} - rSU_S + rU, \quad 0 < S < \infty,\ 0 < \tau \leq T, \tag{2.6}$$

with the terminal and boundary conditions

$$\begin{aligned} U(S,0) &= \max(0, S - E), \\ U(0,\tau) &= 0, \\ \lim_{S\to\infty} \frac{U(S,\tau)}{S - Ee^{-r\tau}} &= 1. \end{aligned} \tag{2.7}$$

In the sequel we consider the fourth-order semidiscretization of the problem (2.6), (2.7) in the form

$$\frac{du}{d\tau} = M(\tau)u(\tau), \quad 0 < \tau \leq T, \tag{2.8}$$

with initial conditions

$$u(0) = [u_1(0), \ldots, u_{N-1}(0)]^\top, u_i(0) = \max(S_i - E, 0),\ 1 \leq i \leq N-1. \tag{2.9}$$

Here $M(\tau)$ is given by

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 0 & \cdots & \cdots & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & \ddots & \ddots & \vdots \\
\alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \xi_3 & 0 & \ddots & \vdots \\
0 & \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \xi_4 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \cdots & 0 & \alpha_{N-3} & \beta_{N-3} & \gamma_{N-3} & \delta_{N-3} & \xi_{N-3} \\
\vdots & \cdots & \cdots & 0 & a_{N-2N-4} & a_{N-2N-3} & a_{N-2N-2} & a_{N-2N-1} \\
0 & \cdots & \cdots & 0 & a_{N-1N-4} & a_{N-1N-3} & a_{N-1N-2} & a_{N-1N-1}
\end{bmatrix}, \quad (2.10)$$

where the values of the entries are:

$$\begin{aligned}
\alpha_i = \alpha_i(\tau) &= -\frac{\sigma_i^2 S_i^2}{24h^2} + \frac{rS_i}{12h}, \\
\beta_i = \beta_i(\tau) &= \frac{2\sigma_i^2 S_i^2}{3h^2} - \frac{2rS_i}{3h}, \\
\gamma_i = \gamma_i(\tau) &= -\frac{15\sigma_i^2 S_i^2}{12h^2} - r, \\
\delta_i = \delta_i(\tau) &= \frac{2\sigma_i^2 S_i^2}{3h^2} + \frac{2rS_i}{3h}, \\
\xi_i = \xi_i(\tau) &= -\frac{\sigma_i^2 S_i^2}{24h^2} - \frac{rS_i}{12h},
\end{aligned} \quad (2.11)$$

where $\sigma_i = \sigma(U_{SS}(S_i, \tau))$, and the following expressions for the nonzero entries of the first, second and last two rows

$$\begin{aligned}
a_{11} &= \gamma_1 + 10\alpha_1 + 4\beta_1, & a_{12} &= \delta_1 - 20\alpha_1 - 6\beta_1, \\
a_{13} &= \xi_1 + 15\alpha_1 + 4\beta_1, & a_{14} &= -4\alpha_1 - \beta_1, \\
a_{21} &= \beta_2 + 4\alpha_2, & a_{22} &= \gamma_2 - 6\alpha_2, \\
a_{23} &= \delta_2 + 4\alpha_2, & a_{24} &= \xi_2 - \alpha_2, \\
a_{N-2N-4} &= \alpha_{N-2} - \xi_{N-2}, & a_{N-2N-3} &= \beta_{N-2} + 4\xi_{N-2}, \\
a_{N-2N-2} &= \gamma_{N-2} - 6\xi_{N-2}, & a_{N-2N-1} &= \delta_{N-2} + 4\xi_{N-2}, \\
a_{N-1N-4} &= -\delta_{N-1} - 4\xi_{N-1}, & a_{N-1N-3} &= \alpha_{N-1} + 4\delta_{N-1} + 15\xi_{N-1}, \\
a_{N-1N-2} &= \beta_{N-1} - 6\delta_{N-1} - 20\xi_{N-1}, & a_{N-1N-1} &= \gamma_{N-1} + 4\delta_{N-1} + 10\xi_{N-1}.
\end{aligned}$$

Remark 3. *We consider the fully nonlinear problem without any linearization process.*

After using the Euler method the numerical solution will be

$$u(\tau) = \left[\prod_{m=\ell-1}^{m=0} (I + kM(mk)) \right] u(0), \tag{2.12}$$

where $k = \triangle\tau,\ \ell k = \tau$ and

$$\sigma_i^2 = \sigma_0^2(1 + \Psi_i(\tau)), \tag{2.13}$$

$$\Psi_i(mk) = \Psi\left(e^{mkr} a^2 S_i^2 \left(\frac{-u_{i-2} + 16u_{i-1} + 16u_{i+1} - u_{i+2}}{12h^2} \right) \right).$$

Here instead of solving numerically the singular ODE

$$\Psi'(A) = \frac{\Psi(A) + 1}{2\sqrt{A\Psi(A)} - A}, \quad A \neq 0,\ \Psi(0) = 0, \tag{2.14}$$

we propose to use the following theorem presented recently in [6]:

Theorem 1. *The nonlinear volatility correction function* Ψ, *unique solution of* (2.14) *satisfies the following properties:*
(i) Ψ *is implicitly defined by*

$$A = \left(-\frac{\operatorname{arcsinh}\sqrt{(\Psi)}}{\sqrt{\Psi + 1}} + \sqrt{\Psi} \right)^2,\ if\ \Psi > 0, \tag{2.15}$$

$$A = -\left(\frac{\arcsin\sqrt{(-\Psi)}}{\sqrt{\Psi + 1}} - \sqrt{-\Psi} \right)^2,\ if\ 0 > \Psi > -1. \tag{2.16}$$

(ii) Ψ *is a one to one increasing function mapping the real line onto the interval* $]-1, +\infty[$.

2.3.1 Classical finite difference schemes

We consider the nonlinear Black-Scholes model (2.4) with the volatility (2.2) as proposed in [8]. In order to transform problem (2.2) into a convection-diffusion problem, we use the following transformation:

$$x(S) = \ln\frac{S}{E}, \quad t(r) = \frac{1}{2}\sigma_0^2(T - t), \quad u = e^{-x}\frac{V}{E}.$$

Then (2.2) may be rewritten in the following form

$$u_t - \left(1 + \Phi[e^{(Kt+x)} a^2 E(u_{xx} + u_x)]\right)(u_{xx} - u_x) - Ku_x = 0, \tag{2.17}$$

where $x \in (-\infty, \infty)$, $0 \le t \le T = \frac{\sigma_0^2 T}{2}$, $K = \frac{2r}{\sigma_0^2}$. The rewritten problem (2.17) has the following boundary and initial conditions:

$$u(x,0) = u_0(x) = \max(1 - e^{-x}, 0),$$
$$u(x,t) = 0 \quad (x \to -\infty), \tag{2.18}$$
$$u(x,t) \sim 1 \quad (x \to +\infty).$$

Here we will consider both standard difference schemes and compact schemes, derived by Rigal [22]. All considered difference schemes have two time levels. Let A^n and B^n be discretization matrices

$$A^n = [a_{-1}, a_0, a_1], \quad B^n = [b_{-2}, b_{-1}, b_0, b_1, b_2]$$

then the schemes may be rewritten in the following form:

$$A^n U^{n+1} = B^n U^n. \tag{2.19}$$

The matrix A^n is tridiagonal and the obtained linear systems may be solved using the Thomas algorithm [25]. Assume that

$$\sum_{i=-1}^{1} a_i = \sum_{i=-2}^{2} b_i = 1.$$

The nonlinearity is expressed explicit in all the schemes. Let us define the volatility correction in the following way:

$$\sigma_i = \Psi \left[\exp\left(Knk + x_i\right) a^2 E \left(\frac{U_{i-2}^n - 2U_i^n + U_{i+2}^n}{4h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h} \right) \right]. \tag{2.20}$$

This formula gives an explicit discretization of nonlinearity and uses a special stencil for the second derivative (step $2h$ instead of h).

Another problem lies in the initial condition for u_0, as it is nondifferentiable in the point $x = 0$. Oosterlee et al. [21] solved this problem of reduced accuracy and proposed *a grid stretching technique*, which is based on an idea of placing more points in the neighborhood of the nondifferentiable payment condition. We use interpolation of high order to smooth the initial data but only with approximation (2.20) useful results can be obtained.

The Figure 2.1 shows solutions of equations (2.15), (2.16) (on the left), and of the ODE (2.14) (on the right) and their spline interpolation. It is easily seen that figures show indistinguishable results. Results are obtained using the standard Matlab routine `fsolve`.

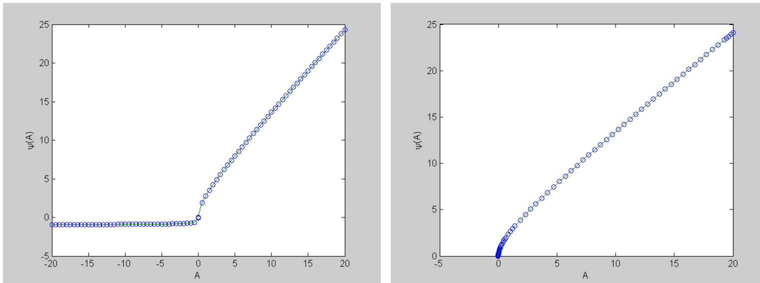

Figure 2.1: *Solutions of equations (2.15), (2.16) and ODE (2.14).*

Let us introduce the notation: $\lambda = -(1+K)$ - the linear part of the coefficient of the convection term in (2.17); $\alpha = \frac{\lambda h}{2}$ - the Reynolds number; $r = \frac{k}{h^2}$ - the parabolic mesh ratio; $\mu = \frac{k}{h}$ - the hyperbolic mesh ratio.

In the following we consider some classical finite difference schemes and study their properties for the linear case but we will investigate these properties for the nonlinear case $a > 0$ further.

Forward-Time Central-Space explicit scheme (FTCS)
This scheme is given by

$$\begin{aligned} &a_{-1} = 0, && a_0 = 1, && a_1 = 0, \\ &b_{-1} = r - \frac{\mu}{2}(\sigma_i - \lambda), && b_0 = 1 - 2r - \frac{r}{2}\sigma_i, && b_1 = r + \frac{\mu}{2}(\sigma_i - \lambda), \\ & && b_{-2} = b_2 = \frac{r}{4}\sigma_i. && \end{aligned}$$

It is of order (1,2), with a very strict stability condition:

$$r \le \frac{1}{2}. \tag{2.21}$$

The condition

$$|\alpha| \le 1 \tag{2.22}$$

must be satisfied to avoid oscillations.

Backward-Time Central-Space semi-explicit scheme (BTCS)
This scheme, with an explicit treatment of the nonlinearity, is given by

$$a_{-1} = \frac{\lambda}{2}\mu - r, \quad a_0 = 1 + 2r, \quad a_1 = -\frac{\lambda}{2}\mu - r,$$
$$b_{-2} = \frac{r}{4}\sigma_i, \quad b_{-1} = -\frac{1}{2}\mu\sigma_i, \quad b_0 = 1 - \frac{r}{2}\sigma_i, \quad b_1 = \frac{1}{2}\mu\sigma_i, \quad b_2 = \frac{r}{4}\sigma_i.$$

It is of order (1,2). It is unconditionally stable and if (2.22) is satisfied, then it is non-oscillatory.

Crank-Nicolson (CN)
This scheme, with an explicit treatment of the nonlinearity, is given by

$$\begin{aligned} a_{-1} &= \left(-\frac{r}{2}+\frac{\mu}{4}\right)\sigma_i - \frac{r}{2} - \frac{\lambda}{4}\mu, \quad & b_{-1} &= \left(\frac{r}{2}-\frac{\mu}{4}\right)\sigma_i + \frac{r}{2} + \frac{\lambda}{4}\mu, \\ a_0 &= 1 + r(1+\sigma_i), \quad & b_0 &= 1 - r(1+\sigma_i), \\ a_1 &= \left(-\frac{r}{2}-\frac{\mu}{4}\right)\sigma_i - \frac{r}{2} + \frac{\lambda}{4}\mu, \quad & b_1 &= \left(\frac{r}{2}+\frac{\mu}{4}\right)\sigma_i + \frac{r}{2} - \frac{\lambda}{4}\mu, \end{aligned}$$

and $b_{-2} = b_2 = 0$. It is of order (2,2) and unconditionally stable.

2.3.2 Compact schemes of higher order

In [22] Rigal introduced several finite difference schemes (FDS) for linear convection-diffusion problems. We will consider only two and apply them to problem (2.17). These schemes are both compact two-level schemes of order (2,4) in the linear case. The nonlinearity is treated semi-implicitly as in the previous subsection.

In these methods several propositions were made. The class of two-level three-point schemes of order (2,4) is defined in the following way:

$$a_{-1}v_{j-1}^{n+1} + a_0 v_j^{n+1} + a_1 v_{j+1}^{n+1} = b_{-1}v_{j-1}^{n} + b_0 v_j^{n} + b_1 v_{j+1}^{n}. \tag{2.23}$$

Matrices A and B are positive and all their entries are positive too. For the FDS (2.23) to be positive matrix $A^{-1}B$ should be positive too. A and B are tridiagonal matrices with diagonals $[a_{-1}, a_0, a_1]$ and $[b_{-1}, b_0, b_1]$ respectively.

For the general construction and description of properties of fourth-order schemes we should refer to two lemmas correlative with two-level three-point schemes (2.23) [22]. Consider the model diffusion-convection problem $\mathfrak{P}$:

$$\partial_t u = \partial_x^2 u - \lambda \partial_x u + f = Au + f \quad \text{in }]0,1[\times[0,T[.$$

We assume that

$$\sum_{i=-1}^{1} a_i = \sum_{i=-1}^{1} b_i = 1,$$

which can always be obtained after a possible normalization of the coefficients in consistent schemes.

Lemma 1. *The FDS* (2.23) *is stable iff the coefficients* a_i, b_i *fulfill*

$$(a_1 - a_{-1})^2 - (b_1 - b_{-1})^2 > a_1 + a_{-1} - b_1 - b_{-1}, \tag{2.24}$$

$$(a_1 - a_{-1})^2 - (b_1 - b_{-1})^2 > a_1 + a_{-1} - b_1 - b_{-1}. \tag{2.25}$$

Lemma 2. *The FDS* (2.23) *is non-oscillatory if the coefficients* a_i, b_i *fulfill*

$$(a_1 - b_1)(a_{-1} - b_{-1}) \geq 0. \tag{2.26}$$

The general two-level three-point scheme ($\mathbf{P_h}$) is defined in the following way:

$$(1+C)D_t v_j^n = \left(\frac{1}{2} + A_1\right) D_+D_- v_j^n + \left(\frac{1}{2} + A_2\right) D_+D_- v_j^{n+1} \tag{2.27}$$

$$-\lambda\left(\frac{1}{2} + B_1\right) D_0 v_j^n - \lambda\left(\frac{1}{2} + B_2\right) D_0 v_j^{n+1},$$

where A_i, B_i, C are real constants chosen in the way to eliminate lower terms in the truncation error. D_t, D_0, D_+, D_- are the basic difference operators

$$\begin{aligned} D_t v_j^n &= \frac{v_j^{n+1} - v_j^n}{\triangle t}, \quad & D_0 v_j^n &= \frac{v_{j+1}^n - v_{j-1}^n}{2h}, \\ D_+ v_j^n &= \frac{v_{j+1}^n - v_j^n}{h}, \quad & D_- v_j^n &= \frac{v_j^n - v_{j-1}^n}{h}. \end{aligned} \tag{2.28}$$

In order to get the truncation error, we need to apply (P_h) to u, smooth enough solution of $(\mathfrak{P}_0)$, the homogeneous problem associated with $(\mathfrak{P})$:

$$\begin{aligned} E_u(\triangle t, h) &= (1+C)D_t u(x_j, t_n) \\ &- \left(\frac{1}{2} + A_2\right) D_+D_- u(x_j, t_{n+1}) \\ &- \left(\frac{1}{2} + A_1\right) D_+D_- u(x_j, t_n) \\ &-\lambda\left(\frac{1}{2} + B_2\right) D_0 u(x_j, t_{n+1}) \\ &-\lambda\left(\frac{1}{2} + B_1\right) D_0 u(x_j, t_n). \end{aligned} \tag{2.29}$$

Here $u(x,t)$ satisfies

$$\partial_t u + \lambda \partial_x u = \partial_x^2 u. \tag{2.30}$$

By decomposing each term in (2.29) we get

$$E_u(\triangle t, h) = \sum_{j=1}^{6} e_j \partial_x^j u + HOD \;\; \text{(higher order derivatives)}, \tag{2.31}$$

where

$$e_1 = \lambda(B_1 + B_2 - C), \tag{2.32}$$

$$e_2 = C - A_1 - A_2 - \lambda^2 \triangle t \left(B_2 - \frac{C}{2} \right), \tag{2.33}$$

$$e_3 = \lambda \left[\triangle t (A_2 + B_2 - C) + \frac{h^2}{6}(1 + B_1 + B_2) \right. \tag{2.34}$$
$$\left. + \lambda^2 \frac{\triangle t^2}{6} \left[\frac{1}{2} + 3B_2 - C \right] \right],$$

$$e_4 = \triangle t \left(\frac{C}{2} - A_2 \right) - (1 + A_1 + A_2) \frac{h^2}{12} + \lambda^2 \frac{\triangle t^2}{2} \left(C - \frac{1}{2} - A_2 - 2B_2 \right) \tag{2.35}$$
$$- \left(\frac{1}{2} + B_2 \right) \lambda^2 \frac{h^2 \triangle t}{6} + \lambda^4 \frac{\triangle t^3}{24}(C - 4B_2 - 1),$$

$$e_5 = \lambda \left[\frac{\triangle t^2}{2} \left(2A_2 + B_2 + \frac{1}{2} - C \right) + \frac{h^4}{120}(1 + B_1 + B_2) \right. \tag{2.36}$$
$$\left. + \frac{h^2 \triangle t}{12} \left(\frac{3}{2} + A_2 + 2B_2 \right) \right],$$

$$e_6 = \frac{\triangle t^2}{2} \left(C - 3A_2 - \frac{1}{2} \right) - (1 + A_1 + A_2) \frac{h^4}{360} - \frac{h^2 \triangle t}{12} \left(\frac{1}{2} + A_2 \right). \tag{2.37}$$

The class of R3 schemes is defined by

$$e_1 = e_2 = e_3 = 0, \tag{2.38}$$

$$E_{R3} = O(\triangle t^2 + h^4), \tag{2.39}$$

that means that e_1, e_2, e_3 and the terms of order less than (2,4) in e_4 should dissappear.

We take $C = 0$ and prescribing (2.38) we express A_1, A_2, B_1 as a function of B_2 :

$$B_1 = -B_2,$$

$$A_2 = -\frac{\lambda^2 \triangle t}{12} - \frac{1}{6r} - B_2 \left(1 + \frac{\lambda^2 \triangle t}{2} \right), \tag{2.40}$$

$$A_1 = \frac{\lambda^2 \triangle t}{12} + \frac{1}{6r} + B_2 \left(1 - \frac{\lambda^2 \triangle t}{2}\right).$$

B_2 must be chosen in such way that

$$e_4 = \frac{h^2}{12} - \frac{\lambda^2 \triangle t^2}{6} + \frac{B_2}{12}[-\lambda^2 h^2 \triangle t + 12\triangle t + \lambda^4 \triangle t^3] \tag{2.41}$$

must be of order (2,4).

R3A scheme

In this scheme the choice for B_2 is

$$B_2 = -\frac{1}{12r}, \tag{2.42}$$

which eliminates in e_4 the only terms depending on $\triangle t$ and h^2. Hence,

$$e_4 = -\frac{\lambda^2 \triangle t^2}{6} + \frac{\lambda^2 h^4}{144} - \frac{\lambda^4 h^2 \triangle t^2}{144}.$$

Replacing B_2 in expressions (2.40) of A_1, A_2 the coefficients will be written in the following way:

$$\begin{aligned}
a_{-1} &= \left(\frac{1}{12} - \frac{r}{2}\right)(1+\alpha) - \frac{\alpha^2 r}{6} + \frac{\alpha^2 r^2}{3}, \\
a_0 &= \frac{5}{6} + r + \frac{\alpha^2 r}{3} - \frac{2\alpha^2 r^2}{3}, \\
a_1 &= \left(\frac{1}{12} - \frac{r}{2}\right)(1-\alpha) - \frac{\alpha^2 r}{6} + \frac{\alpha^2 r^2}{3}, \\
b_{-2} &= \frac{r}{4}\sigma_i, \\
b_{-1} &= \left(\frac{1}{12} + \frac{r}{2}\right)(1+\alpha) + \frac{\alpha^2 r}{6} + \frac{\alpha^2 r^2}{6} - \frac{1}{2}\mu\sigma_i, \\
b_0 &= \frac{5}{6} - r - \frac{\alpha^2 r}{3} - \frac{2\alpha^2 r^2}{3} - \frac{r}{2}\sigma_i, \\
b_1 &= \left(\frac{1}{12} + \frac{r}{2}\right)(1-\alpha) - \frac{\alpha^2 r}{6} + \frac{\alpha^2 r^2}{3} + \frac{1}{2}\mu\sigma_i, \\
b_2 &= \frac{r}{4}\sigma_i.
\end{aligned}$$

It is stable in the linear case $\sigma_i = 0$ if

$$r \leq \frac{1}{\sqrt{2}|\alpha|}, \tag{2.43}$$

cf. [22]. If α is arbitrary, then this scheme is non-oscillatory.

R3B scheme
B_2 for this scheme is defined in the following form:

$$B_2 = -\frac{1}{12r} - \frac{\lambda^2 \triangle t}{12}. \tag{2.44}$$

From (2.40) we have

$$B_1 = -B_2,$$
$$A_1 = \frac{1}{12r} + \frac{\lambda^2 h^2}{24} + \frac{\lambda^4 \triangle t^2}{24},$$
$$A_2 = -\frac{1}{12r} + \frac{\lambda^2 h^2}{24} + \frac{\lambda^4 \triangle t^2}{24}.$$

The coefficients for this scheme are the following:

$$\begin{aligned}
a_{-1} &= \left(\frac{1}{12} - \frac{r}{2}\right)(1+\alpha) - \frac{\alpha^2 r}{6} + \frac{\alpha^2 r^2}{3} - \frac{2\alpha^4 r^3}{3}, \\
a_0 &= \frac{5}{6} + r + \frac{\alpha^2 r}{3} + \frac{4\alpha^4 r^3}{3}, \\
a_1 &= \left(\frac{1}{12} - \frac{r}{2}\right)(1-\alpha) - \frac{\alpha^2 r}{6} - \frac{\alpha^3 r^2}{3} - \frac{2\alpha^4 r^3}{3}, \\
b_{-2} &= \frac{r}{4}\sigma_i, \\
b_{-1} &= \left(\frac{1}{12} + \frac{r}{2}\right)(1+\alpha) + \frac{\alpha^2 r}{6} + \frac{\alpha^3 r^2}{3} + \frac{2\alpha^4 r^3}{3} - \left(\frac{r}{4} + \frac{1}{2}\mu\right)\sigma_i, \\
b_0 &= \frac{5}{6} - r - \frac{\alpha^2 r}{3} - \frac{4\alpha^4 r^3}{3} - 2r\sigma_i, \\
b_1 &= \left(\frac{1}{12} - \frac{r}{2}\right)(1+\alpha) + \frac{\alpha^2 r}{6} - \frac{\alpha^3 r^2}{3} + \frac{2\alpha^4 r^3}{3} - \left(\frac{r}{4} - \frac{1}{2}\mu\right)\sigma_i, \\
b_2 &= \frac{r}{4}\sigma_i.
\end{aligned}$$

It is unconditionally stable and non-oscillatory in the linear case $\sigma_i = 0$ [22].

2.4 The fixed domain transformation

Compact schemes which many authors have applied to the Black-Scholes equation with transaction costs have one severe disadvantage: these schemes cannot be generalized to multi-dimensional problems, and are (directly) applicable only to European type options. However, with the fixed domain transformation of Ševčovič [2], [23] we will overcome this shortcoming.

We consider the Black-Scholes equation

$$0 = V_t + \frac{1}{2}\tilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r - q)SV_S - rV, \qquad (2.45)$$

where q - dividend yield, is constant, $S > 0$, $t \in (0, T)$. This equation is supplied with the following terminal and boundary conditions:

$$\begin{aligned} &V(S,T) = (S-K)^+ && \text{for } 0 \le S \le S_f(T), \\ &V(0,t) = 0 && \text{for } 0 \le t \le T, \\ &V(S_f(t),t) = S_f(t) - K && \text{for } 0 \le t \le T, \\ &V_S(S_f(t),t) = 1 && \text{for } 0 \le t \le T, \\ &S_f(T) = \max(K, rK/q). \end{aligned} \qquad (2.46)$$

For simplicity we assume that $r > q$ and we have $S_f(T) = rK/q$ for the American call option.

Equation (2.45) subject to (2.46) is a backward-in-time parabolic free boundary problem. To solve this free boundary problem numerically, many different methods are developed, e.g. the standard method consists in the reformulation to a linear complementary problem (LCP) and solution by a projected SOR method of Cryer [7]. Alternatively, penalty and front-fixing methods were developed (e.g in [11], [20]). A disadvantage of these methods is the change of the underlying model. A different approach [13] is based on a recursive calculation of the early exercise boundary, estimating the boundary by Richardson interpolation. Explicit boundary tracking algorithms are e.g. a finite difference bisection scheme [16] or the *front-tracking strategy of Han and Wu* [12].

In this thesis we consider the approach of Ševčovič [23]. We want to simplify the numerical solution of (2.45), (2.46) for American call options and get rid of the (explicit) appearance of the free boundary. To do this, we need to transform the problem into a problem posed on a fixed, but unbounded domain additionally to the forward transformation in time. Then, the domain does not depend on the free boundary $S_f(t)$ anymore. All we need is to calculate an algebraic constraint equation for the position of the free boundary. Let's make the following substitution:

$$\tau = T - t,\ x = \ln\left(\frac{\varrho(\tau)}{S}\right) \Leftrightarrow S = e^{-x}\varrho(\tau),\ \varrho(\tau) = S_f(T - \tau),$$

such that $x \in \mathbb{R}^+$ and $\tau \in [0, T]$.

The constructed (synthetic) portfolio will be the following:

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t). \tag{2.47}$$

After differentiating this portfolio with respect to x and τ and substituting the result into (2.45) we get

$$0 = \Pi_\tau + \left(b(\tau) - \frac{\tilde{\sigma}^2}{2}\right)\Pi_x - \frac{1}{2}\partial_x(\tilde{\sigma}^2\Pi_x) + r\Pi, \tag{2.48}$$

defined on $x \in \mathbb{R}^+,\ 0 \le \tau \le T$, where the coefficient $b(\tau)$ is given by

$$b(\tau) = \frac{\varrho'}{\varrho}(\tau) + r - q.$$

The boundary and initial conditions from (2.46) transform into:

$$\Pi(x,0) = V(S,T) - SV_S(S,T) = \begin{cases} -K \text{ for } S > K \ \Leftrightarrow\ x < \ln\dfrac{\varrho(0)}{K} \\ 0 \text{ otherwise} \end{cases}, \tag{2.49}$$

$$\begin{aligned} \Pi(x,\tau) &= 0 \qquad \text{as } x \to \infty,\ 0 \le \tau \le T, \\ \Pi(0,\tau) &= -K \qquad \text{for } 0 \le \tau \le T. \end{aligned} \tag{2.50}$$

With the assumption $r \ge q$ we obtain

$$\varrho(\tau) = \frac{1}{2q}\tilde{\sigma}^2\Pi_x(0,\tau) + \frac{rK}{q} \quad \text{with } \varrho(0) = \frac{rK}{q}, \tag{2.51}$$

where $0 \le \tau \le T$ and the modified volatility function becomes

$$\tilde{\sigma}^2 = \sigma^2\left(1 + \Psi(e^{r\tau}a^2\Pi_x)\right). \tag{2.52}$$

Our transformed problem (2.48) subject to (2.49)-(2.51) with the volatility function (2.52) can be solved e.g. by the split-step finite-difference method proposed by Ševčovič [23].

After using this method and solving the transformed problem, we can calculate the value of the American call option $V(S,t)$ by transforming (2.47) back to the original variables. Since we know that

$$\frac{\Pi(x,\tau)}{S^2} = \frac{V(S,t)}{S^2} - \frac{V_S(S,t)}{S} = \partial_S\left(-\frac{V(S,t)}{S}\right),$$

we integrate the above equation from S to $S_f(t)$ with the boundary condition $V(S_f(t),t) = S_f(t) - K$ and we get

$$V(S, T-\tau) = \frac{S}{\varrho(\tau)} \left(\varrho(\tau) - K + \int_{0}^{\ln \frac{\varrho(\tau)}{S}} e^x \Pi(x,\tau) dx \right). \tag{2.53}$$

Thus, (2.53) yields the price of the American call option $V(S,t)$ in the presence (and absence) of transaction costs.

2.5 The method of Liao and Khaliq

Liao and Khaliq [19] proposed a new high order compact scheme (HOC). Let's suppose we have the following one-dimensional time dependent convection-diffusion equation:

$$u_t = \beta u_{xx} + \lambda u_x, \tag{2.54}$$

where β and λ are constants. This equation is transformed into a system of two equations. The following new unknown function is introduced:

$$v(x,t) = u_x(x,t), \tag{2.55}$$

so

$$u_t = \beta u_{xx} + \lambda v. \tag{2.56}$$

We can obtain the same results even if there is a reaction term $f(u)$ in the equation (2.54):

$$u_t = \beta u_{xx} + \lambda v + f(u) \tag{2.57}$$

$$v_t = \beta v_{xx} + \lambda u_{xx} + \frac{\partial f}{\partial u} v. \tag{2.58}$$

For the sake of completeness we need to write the initial and boundary conditions. For $u(x,t)$ it will be

$$u(x,0) = u_0(x),$$

$$u(0,t) = b_0(t),$$

$$u(1,t) = b_1(t).$$

For $v(x,t)$ the condition will be a derivative of $u(x,t)$ with respect to x then letting $t \to 0$:

$$v(x,0) = u_0'(x).$$

Remark 4. *Usually we cannot derive the initial condition for $v(x,t)$. Thus we need to propose some numerical approximation. In [19] a compact fourth order numerical approximation is proposed to approximate $v(0,t)$ and $v(1,t)$.*

We have the central difference operator $\triangle_x^0$:

$$\triangle_x^0 u_j = u_{j+1} - u_{j-1}. \tag{2.59}$$

Suppose the grid is uniform, i.e. N sub-intervals form the interval $[0,1]$ and $h = \frac{1}{N}$. We can write a second order approximation

$$v(h,t) = \frac{\partial u}{\partial x}(h,t) \approx \frac{u(2h,t) - u(0,t)}{2h} = \frac{\triangle_x^0}{2h} u(h,t). \tag{2.60}$$

It can be improved to fourth order if $\triangle_x^0$ is replaced by $\frac{\triangle_x^0}{1+\frac{1}{6}\triangle_x^2}$:

$$v(h,t) = \frac{\triangle_x^0}{2h(1+\frac{1}{6}\triangle_x^2)u(h,t)}. \tag{2.61}$$

And we can get a fourth order approximation:

$$v(0,t) = \frac{3}{h}\left(u(2h,t) - u(0,t)\right) - 4v(h,t) - v(2h,t). \tag{2.62}$$

Hence, we can approximate the right boundary condition of v at $x = 1$ as

$$v(1,t) = \frac{3}{h}\left(u(1-h,t) - u(1-2h,t)\right) - 4v(1-h,t) - v(1-2h,t). \tag{2.63}$$

Let us consider a more general system:

$$u_t = \beta u_{xx} + f(u,v), \tag{2.64}$$

$$v_t = \lambda u_{xx} + \beta v_{xx} + g(u,v), \tag{2.65}$$

where the term λv is included in the general function $f(u,v)$ and there is only one diffusion term βu_{xx} in (2.64).

Now we should apply the new method. To start with, we write the Crank-Nicolson scheme:

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} = \frac{1}{2}\left(\frac{\beta}{h^2}\triangle_x^2 u_i^{n+1} + \frac{\beta}{h^2}\triangle_x^2 u_i^n + f_i^{n+1} + f_i^n\right), \tag{2.66}$$

$$\frac{v_i^{n+1} - v_i^n}{\triangle t} = \frac{1}{2}\left(\frac{\lambda}{h^2}\triangle_x^2[u_i^{n+1} + u_i^n] + \frac{\beta}{h^2}\triangle_x^2[v_i^{n+1} + v_i^n] + g_i^{n+1} + g_i^n\right), \tag{2.67}$$

where

$$\begin{aligned} f_i^{n+1} =& f(u_i^{n+1}, v_i^{n+1}), \\ f_i^n =& f(u_i^n, v_i^n), \\ g_i^{n+1} =& g(u_i^{n+1}, v_i^{n+1}), \\ g_i^n =& g(u_i^n, v_i^n), \end{aligned}$$

the standard second order difference operator $\triangle_x^2$ is defined as:

$$\triangle_x^2 u_i = u_{i+1} - 2u_i + u_{i-1}.$$

But $(u_{xx})_i \approx \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i+1})$ gives only the second order approximation to u_{xx}. We can further improve this approximation to the fourth order by using instead the Padé approximation

$$(u_{xx})_i \approx \frac{\triangle_x^2}{h^2(1 + \frac{1}{12}\triangle_x^2)}.$$

We apply this Padé approximation in (2.66)-(2.67), multiply both sides by $1 + \frac{1}{12}\triangle_x^2$ and the new scheme can be rewritten as

$$\left(1 + \frac{\triangle_x^2}{12} - \frac{\beta r_x}{2}\right) u_i^{n+1} = \left(1 + \frac{\triangle_x^2}{12} - \frac{\beta r_x}{2}\right) u_i^n \tag{2.68}$$

$$+\frac{\triangle t}{2}\left(1 + \frac{\triangle_x^2}{12}\right)(f_i^{n+1} + f_i^n),$$

$$\left(1 + \frac{\triangle_x^2}{12} - \frac{\beta r_x}{2}\right) v_i^{n+1} = \left(1 + \frac{\triangle_x^2}{12} - \frac{\beta r_x}{2}\right) v_i^n + \lambda\frac{r_x}{2}\triangle_x^2(u_i^{n+1} + u_i^n) \tag{2.69}$$

$$+\frac{\triangle t}{2}\left(1 + \frac{\triangle_x^2}{12}\right)(g_i^{n+1} + g_i^n),$$

where $r_x = \frac{\triangle t}{h^2}$.

It can be easily showed that the truncation error of (2.68) and (2.69) is $C_1\triangle t^2 + C_2\triangle t^4 + C_3 h^4$. So, the Richardson extrapolation can be used here to improve the approximation to fourth order in time.

Suppose the solutions of (2.68) and (2.69) after k iterations are $u_i^{n+1^{(k)}}$ and $v_i^{n+1^{(k)}}$ respectively. To get $u_i^{n+1^{(k+1)}}$ and $v_i^{n+1^{(k+1)}}$, we first should expand f_i^{n+1} in the following form:

$$f(u_i^{n+1}, v_i^{n+1}) = f(u_i^{n+1^{(k)}}, v_i^{n+1^{(k)}}) + \frac{\partial f}{\partial u}(u_i^{n+1^{(k)}}, v_i^{n+1^{(k)}})(u_i^{n+1} - u_i^{n+1^{(k)}}) \tag{2.70}$$

and insert it into (2.68), then solve the following equation for $u_i^{n+1^{(k+1)}}$

$$\left(1+\frac{1}{12}\triangle_x^2-\frac{\beta r_x}{2}\triangle_x^2-\frac{\triangle t}{2}(1+\frac{1}{12}\triangle_x^2)\hat{\mathbf{J}}_i^{n+1^{(k)}}\right)u_i^{n+1^{(k+1)}}=$$

$$=\left(1+\frac{1}{12}\triangle_x^2-\frac{\beta r_x}{2}\triangle_x^2\right)u_i^n+$$

$$+\frac{\triangle t}{2}(1+\frac{1}{12}\triangle_x^2)\left(f(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})-\hat{\mathbf{J}}_i^{n+1^{(k)}}u_i^{n+1^{(k)}}+f(u_i^n,v_i^n)\right), \quad (2.71)$$

where $\hat{\mathbf{J}}_i^{n+1^{(k)}}=\frac{\partial f}{\partial u}(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})$.

Once we have found $u_i^{n+1^{(k+1)}}$, we expand g_i^{n+1} as

$$g(u_i^{n+1},v_i^{n+1})=g(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})+\frac{\partial g}{\partial u}(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})(u_i^{n+1}-u_i^{n+1^{(k)}}). \quad (2.72)$$

Substituting (2.72) into (2.69) we get:

$$\left(1+\frac{1}{12}\triangle_x^2-\frac{\beta r_x}{2}\delta_x^2-\frac{\triangle t}{2}(1+\frac{1}{12}\triangle_x^2)\tilde{\mathbf{J}}_i^{n+1^{(k)}}\right)v_i^{n+1^{(k+1)}}$$

$$=\left(1+\frac{1}{12}\triangle_x^2-\frac{\beta r_x}{2}\triangle_x^2\right)u_i^n$$

$$+\frac{\triangle t}{2}(1+\frac{1}{12}\triangle_x^2)\left(f(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})-\tilde{\mathbf{J}}_i^{n+1^{(k)}}u_i^{n+1^{(k)}}+f(u_i^n,v_i^n)\right), \quad (2.73)$$

where $\tilde{\mathbf{J}}_i^{n+1^{(k)}}=\frac{\partial g}{\partial u}(u_i^{n+1^{(k)}},v_i^{n+1^{(k)}})$. We then solve (2.73) for $v_i^{n+1^{(k+1)}}$. The two steps are repeated alternatively until convergence occurs.

In this section we considered several finite difference schemes and their application to the Barles-Soner volatility function (2.2). Instead of solving the singular differential equation (1.7) we used properties of $\Psi=\Psi(A)$ (2.15), (2.16) described recently in [6]. Calculating equations (2.15) and (2.16) and comparing them to the solution of the ODE (2.14) showed indistinguishable results. This proves numerically the theory of Company, Navarro, Pintos and Ponsoda [6].

Chapter 3

Numerical solution

Many researchers tried to find solution of Black-Scholes equation but there are not so many general-form solutions and authors develop numerical methods for solving this equation.

Many of found methods can only be applied to European options, like the method of Liao and Khaliq [19] and methods derived by Rigal [22]. In case of American options another strategy is needed. First the equation should be transformed into the heat equation. Then domain should modified into semiunbounded with a free boundary. Then the heat equation is solved on this domain.

Exact analytical formulas for the free bundary $S_f(t)$ in (2.3) with conditions (2.5) is not known, but there exist several deductions of approximate formulas for American option estimation in linear case. Recently Ševčovič in [23] proposed a new method of transformation of the free boundary problem for the early exercise boundary location into deduction of time dependent nonlinear parabolic equation on a fixed domain.

In this chapter we will try to apply the method of Liao and Khaliq to a nonlinear Black-Scholes equation in case of American options. In their work [19] Liao and Khaliq offered an unconditionally stable compact finite-difference scheme of fourth order both in space and time. In numerical examples they show the application of the method to the linear convection-diffusion equation and nonlinear equation of Black-Scholes equation in case of European options. In this chapter we will apply this method to nonlinear Black-Scholes equation in case of American options.

3.1 American options

We consider equation (2.3) with boundary conditions (2.5) and the volatility function (2.2). This equation is a backward in time free boundary problem. We need to transform the grid into a fixed but unbounded domain and apply a forward transformation in time.

As was proposed in [23] the following change of variables is introduced:

$$\tau = T - t,\ x = \ln\left(\frac{\varrho(\tau)}{S}\right) \ \Leftrightarrow\ S = e^{-x}\varrho(\tau),\ \varrho(\tau) = S_f(T-t).$$

Building a portfolio

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t),$$

applying the change of variables and the portfolio to our equation, we get the following transformed problem:

$$0 = \Pi_\tau + \left(b(\tau) - \frac{\tilde{\sigma}^2}{2}\right)\Pi_x - \frac{1}{2}\partial_x(\tilde{\sigma}^2\Pi_x) + r\Pi, \quad x \in \mathbb{R}^+,\ 0 \le \tau \le T \tag{3.1}$$

with the volatility function (2.2) and the conditions

$$\Pi(x,0) = \begin{cases} -K & \text{for } x < \ln\dfrac{r}{q}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Pi(x,\tau) = 0 \ \text{ as } x \to \infty, 0 \le \tau \le T, \tag{3.2}$$

$$\Pi(0,\tau) = -K \ \text{ for } 0 \le \tau \le T,$$

and the restriction

$$\varrho(\tau) - \frac{1}{2q}\tilde{\sigma}^2\Pi_x(0,\tau) + \frac{rK}{q} \ \text{ with } \ \varrho(0) = \frac{rK}{q}. \tag{3.3}$$

3.2 Grid

For the grid we refer to the work of Ehrhardt and Ankudinova [2]. We confine the unbounded domain $x \in \mathbb{R}^+$ and $\tau \in [0,T]$ to $x \in (0,R)$ with $R > 0$ large enough. Ševčovič [23] chooses $R = 3$. We also take $h > 0$ - step in space, $k > 0$ - step in time, $x_i = ih,\ i \in [0,N],\ R = Nh,\ \tau_n = nk,\ n \in [0,M],\ T = Mk$.

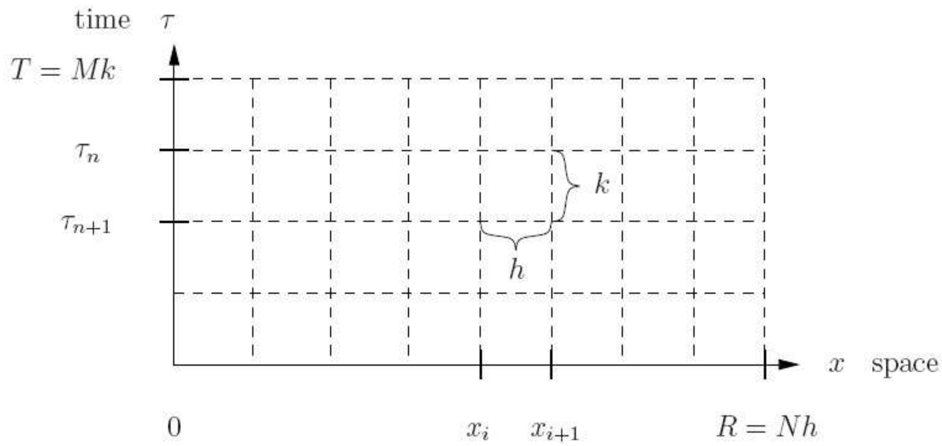

Figure 3.1: *Uniform grid for American options.*

3.3 Free boundary

We consider the free boundary (3.3). Let's take the equation in the point $x = 0$ and approximate the derivative in time by forward differences. We obtain is the following result:

$$\varrho^n = \frac{1}{2q}\sigma(1+\Psi(e^{r\tau_n}a^2 D_h^+\Pi_0^n))D_h^+\Pi_0^n + \frac{rK}{q} \quad \text{with} \varrho^0 = \frac{rK}{q}, \tag{3.4}$$

where $D_h^+\Pi_0^n = \frac{\Pi_1^n - \Pi_0^n}{h}$ is the forward difference coefficient in point $x = 0$.

Fig. 3.2 shows the difference between the free boundary and the asymptotic free boundary taken with exercise date $T = 1$ year, exercise price $K = 10$ and dividend yield $q = 0.05$.

Remark 5. *The dividend yield is a way to calculate how much money is received for every dollar invested. Investors that demand a minimum amount of money from their investment portfolio, may hedge this amount by investing it into assets that pay out relatively high stable dividend yield.*

Sometimes dividends are so frequent that they can be calculated as continuous payments. But usually dividends are paid only several times per year and are considered as discrete. In this case the main problem is how to include discrete dividend payments in Black-Scholes equation.

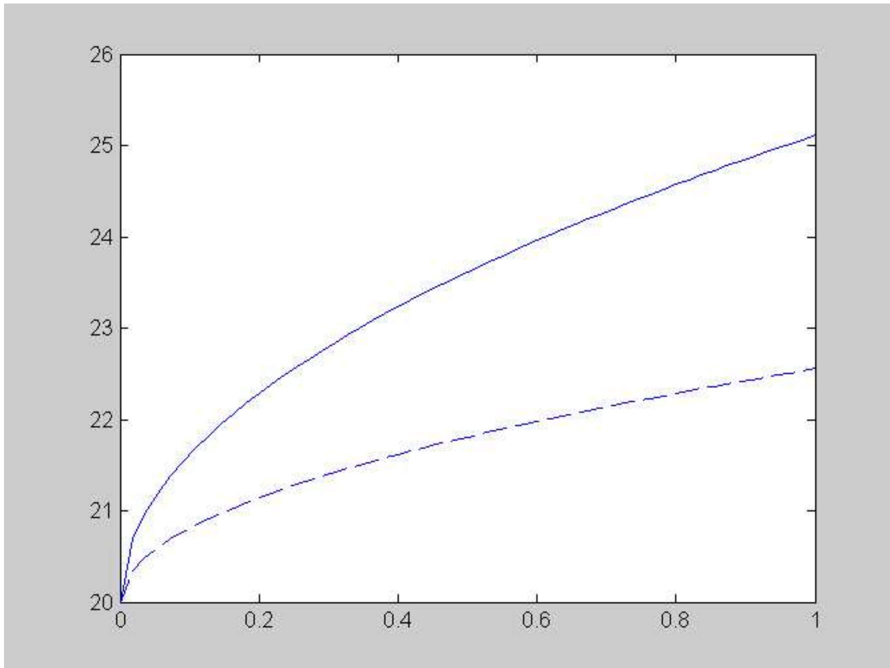

Figure 3.2: *Free boundary (solid) and asymptotic free boundary (dashed) with* $T = 1, K = 10, r = 0.1, q = 0.05$.

3.4 Method of Liao and Khaliq for American Options

Liao and Khaliq in their work [19] proposed an efficient fourth-order numerical algorithm. It is based on the Padé approximation and derived with the help of Richardson extrapolation. The method is described in Section 2.5. This algorithm provides a fast, neat and clear option pricing with transaction costs. It fits best to use it for the nonlinear case of the Black-Scholes equation. In this algorithm a single convection-diffusion equation is split into a system of two reaction-diffusion equations.

The method was applied to the nonlinear Black-Scholes equation in case of European options. To apply the method to American options case we need to combine it with some other method. First, we use the fixed domain transformation described in [2]. Making the grid semi-unbounded and using the substitution proposed by Ševčovič provides us the foundation for further use of the method. In the calculations we use the volatility function of Barles and Soner [3]. And instead of solving the ODE (2.14) we use Theorem 1 (Section 2.3) [6].

After the change of variables, we find the boundary (see Section 3.3.). And using the iteration method described in Section 2.5 we find the price of the option.

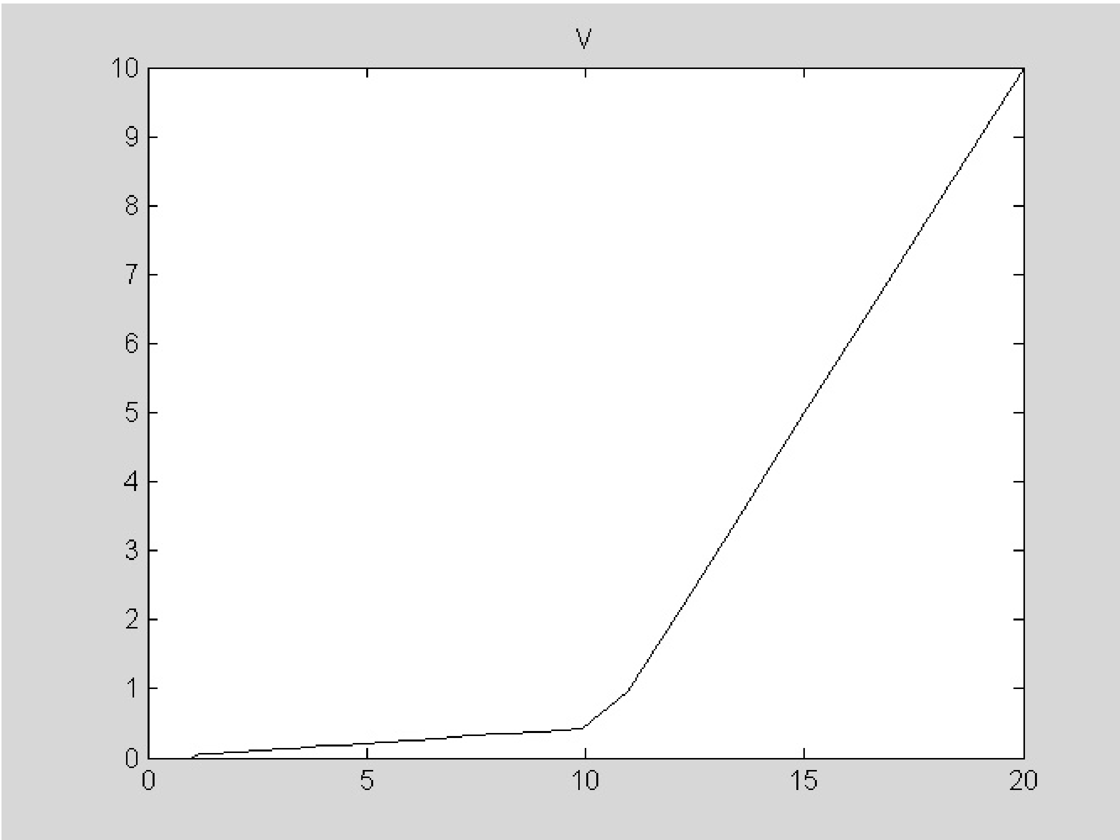

Figure 3.3: *Price of the American option in the presence of transaction costs (in case of nonlinear Black-Scholes equation).*

3.5 Algorithm

The calculations of the price $V(S,t)$ for the American options in the presence of transaction costs lead us to the following algorithm:
Algorithm. Computation of the price $V(S,t)$ for the American option
Input parameters: $\sigma, r, q, K, a, R, T, h, k, M, N, \gamma, \alpha, \beta$
1: solve formulas (2.15), (2.16) for the Barles and Soner volatility model and interpolate the solutions
2: initialize Π^0, ϱ^0, V^T

3: calculate $\Pi^{n,p}$ for each time level iteratively:
3.1: calculate volatility correction for time step τ_n using results from the previous time step
3.2: calculate free boundary for the time step τ_n using results from the previous time steps and volatility correction from the current time step
3.3: calculate the solution of the convective part (equation (2.68)) using tridiagonal matrices
3.4: calculate the solution of the diffusive part (equation (2.69))
4: transform Π into V
5: plot V for each time level and each stock price.

Chapter 4

Conclusions

In this work we made an overview of the nonlinear Black-Scholes equation in the presence of transaction costs in case of American options. Also, some numerical methods were considered.

We began by introducing in Chapter 1 a general overview of the work and some transaction cost models including Leland's model, Barles and Soner model and the Risk Adjustment Pricing Methodology. In Chapter 2 a general introduction to the option pricing terminology was given. Then several high order compact schemes were considered. In order to improve the results and make the results more precise the scheme should be fourth order both in time and space. The needed scheme is proposed in Section 2.5. The method of Liao and Khaliq [19] performs better than other methods described in Section 2.3.

In order to prove that the schemes work good we applied it to the nonlinear Black-Scholes equation with transaction costs in case of American option which was presented in Chapter 3. But before applying it to the equation we needed to make some arrangements. This grid should be made into a fixed semi-unbounded domain and some change of variables should be performed. After that the method could be applied to the American options. In Section 3.4 the numerical results are shown and in Section 3.5 the algorithm of the MatLab program is briefly introduced.

The obtained results provide a possibility for some future research directions, i.e. implementation of (discrete) artificial boundary conditions, cf. [10], since (2.18) is posed on an unbounded domain. Also, the order of the time splitting method can be improved by using e.g. the classic second order splitting method of Strang [24].

Notation

K	exercise price.
S_T	final value of the underlying asset.
r	risk-free rate.
γ	risk-averse factor.
$V,\ V_S,\ V_{SS}$	option price and its derivatives.
$U,\ U_S,\ U_{SS}$	option price and its derivations after the time reversal transformation.
$\Psi(x)$	solution of the nonlinear ODE.
$a_i,\ b_i$	coefficients of the compact schemes.
σ	volatility function.
$\triangle_x^2$	standard second order differential operator.
ρ^n	free boundary.

Bibliography

[1] J. Ankudinova (2008)
The numerical solution of nonlinear Black-Scholes equations. Diploma Thesis, Technische Universität Berlin, Berlin, Germany.

[2] J. Ankudinova, M. Ehrhardt (2008)
Fixed domain transformations and split-step finite difference schemes for Nonlinear Black-Scholes equations for American Options.
Chapter 8 in M. Ehrhardt (ed.)
Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing. Nova Science Publishers, Inc., Hauppauge, NY 11788, pp. 243–273.

[3] G. Barles, H.M. Soner (1998)
Option pricing with transaction costs and a nonlinear Black-Scholes equation. Finance Stochast. 2, 369-397.

[4] F. Black, M.S. Scholes (1973)
The pricing of options and corporate liabilities. J.Political Economy, 81, 637-654.

[5] P. Boyle, T. Vorst (1973)
Option replication in discrete time with transaction costs. Journal of Finance 47, 271-293.

[6] R. Company, E. Navarro, J.R. Pintos, E. Ponsoda (2008)
Numerical solution of linear and nonlinear Black-Scholes option pricing equations. Comput. Math. Appl. 56, 813-821.

[7] E. Cryer (1971)
The solution of a quadratic programming problem using systematic overrelaxation. SIAM J. Control 9, 385–392.

[8] B. Düring (2005)
Black-Scholes Type Equations: Mathematical Analysis, Parameter Iden-

tification and Numerical Solution. Dissertation, Universität Mainz, Germany.

[9] B. Düring, M. Fournié, A. Jüngel (2003)
High-order compact finite difference schemes for a nonlinear Black-Scholes equation. Intern. J. Theor. Appl. Finance 6, 767-789.

[10] M. Ehrhardt, R.E. Mickens (2008)
A fast, stable and accurate numerical method for the Black–Scholes equation of American options. Int. J. Theoret. Appl. Finance 11, 471–501.

[11] P. A. Forsyth, K. R. Vetzal (2002)
Quadratic Convergence for Valuing American Options using a Penalty Method. SIAM J. Sci. Comput. 23, 2095–2122.

[12] H. Han, X. Wu (2003)
A fast numerical method for the Black–Scholes equation of American options. SIAM J. Numer. Anal. 41, 2081–2095.

[13] J.-Z. Huang, M.G. Subrahmanyam, G.G. Yu (1996)
Pricing and Hedging American Options: A Recursive Integration Method. Review of Financial Studies 9, 277–300.

[14] M. Jandačka, D. Ševčovič (2005)
On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile. J. Appl. Math. 3, 235-258.

[15] Y. Kabanov, M. Safarian (1997)
On Leland's strategy of option pricing with transaction costs. Finance Stoch. 1, 239-250.

[16] T. Kim, Y. Kwon (2002)
Finite–Difference Bisection Algorithms for free boundaries of American Call and Put Options. Preprint, Pohang University of Science and Technology, South Korea.

[17] M. Kratka (1998)
No mystery behind the smile. Risk 9, 67-71.

[18] M. Leland. (1985)
Option pricing and replication with Transaction Costs. Journal of Finance 5, 1283-1301.

[19] W. Liao, A.Q.M. Khaliq (2009)
High order compact scheme for solving nonlinear Black-Scholes equation with transaction cost. International Journal of Computer Mathematics, Volume 86 , Issue 6, 1009-1023.

[20] B. F. Nielsen, O. Skavhaug, A. Tveito (2002)
Penalty and front–fixing methods for the numerical solution of American option problems. J. Comp. Finance 5, 69–97.

[21] C. W. Oosterlee, C.C.W. Leentvaar, A. Almendral (2004)
Pricing options with discrete dividends by high order finite differences and grid stretching. ECCOMAS 2004, P. Neittaanmäki et al. (eds.), CD Rom, Jyväskylä, Finland.

[22] A. Rigal (1994)
High Order Difference Schemes for Unsteady One-Dimensional Diffusion-Convection Problems. J. Comp. Phys. 114, 59-76.

[23] D. Ševčovič (2007)
An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear Black-Scholes equation. Canad. Appl. Math. Quarterly 15, 77-79.

[24] G. Strang (1968)
On the construction and comparison of difference schemes. SIAM J. Numer. Anal. 5, 506-517.

[25] L. H. Thomas (1949)
Elliptic problems in linear difference equations over a network. Watson Sci. Comput. Lab. Rept., Columbia University, New York.