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High-order compact methods for the American Option pricing problem

Master's Thesis in Financial Mathematics

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MASTER'S PROGRAMME IN FINANCIAL MATHEMATICS

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Abstract

In this paper, we find numerical solution for the American option pricing problem. The Black-Scholes equation is transformed to the diffusion equation on an unbounded domain. An artificial boundary is introduced to limit this domain. The diffusion operator is discretized by using the different schemes for the interior grid with different boundary conditions. The considered high order compact schemes Method 1 proposed by MacCartin, Method 2 proposed by Tangman and R3-Methods belong to weighted scheme of the approximation for the pure heat equation. Moreover, they coincide and this method is called *the optimal weighted scheme*. This scheme is unconditionally stable and it has the fourth order approximation in space and the second in time. Also, we compare Crank-Nicolson and five-point two-level non-compact stencils with compact method. They show worth numerical result in comparison with high order compact method. The Crank-Nicolson scheme with different boundary conditions is inferior to the non-compact Heun's method. But the Heun's scheme is conditionally stable, which is a big disadvantage for this scheme.

On the right boundary of the truncated computational domain we use the Dirichlet boundary condition. The boundary conditions on the left boundary have influence for the numerical solution of the heat equation. The numerical results show that the combination of high order compact schemes with Han and Wu boundary condition is more accurate to the exact solution which obtained by using the Binomial method with large number of the steps.

Contents

Chapter 1

Introduction

In October 1997 Robert Merton, Myron Scholes were awarded by the "Price of Swedish Bank on the theory of A. Nobel for economics". They derived a classical mathematical formula for options and other derivatives pricing. This formula is well-known as the Black-Scholes formula. It had a great influence on the development of theory and practice of finance. Usage of mathematical methods can't completely describe real market behavior, but it can estimate risk, premium and value of options or derivatives.

The specific character of investigation in financial markets often requires considerably hard conditions to numerical methods. First of all, when using large space steps of the grid we have to use stable implicit schemes. It can avoid using too small time steps, which are typical for explicit schemes. Efficiency is also a wanted property of numerical methods. This property is inherent for schemes consisting of three points in space, which are called *compact schemes*.

There are a lot of methods using compact finite difference schemes for numerical pricing of options and derivatives. In [8] the authors considered the pricing of American and European options. They apply a standard transformation for Black-Scholes equation and obtain the heat equation. To solve this problem they consider the Crandall-Douglas scheme, which is second-order accurate in time and fourth-order accurate in space. If we restrict the size of the time step, this scheme can achieve sixth-order accuracy. The authors compare the Crandall-Douglas scheme with other theta-methods and get result that this scheme induces a smaller error.

To solve the same problem for American options on the unbounded domain Han and Wu [5] introduce an artificial boundary, which separates the domain into bounded and unbounded parts. They show that the accurate solution can be obtained using the artificial boundary conditions. Also it reduces the computational costs since a smaller computational domain can be used.

Tangman, Gopaul, Bhuruth [13] describe a new compact finite difference method, which is an improvement of Han and Wu's algorithm for American options. They use a non-uniform grid. In the area near the spot prices the grid becomes more dense, in the area of non-interest the grid is more coarse. It allows to accurately locate the free boundary. They present the computational results of this method, Han and Wu's method and other known algorithms. The results illustrate that their new method is faster and more accurate.

Often artificial boundaries are incorporated to confine the computational area. Ehrhardt introduces a new type of artificial boundary conditions - discrete TBC (transparent boundary conditions) [3]. He compares this conditions with Mayfield's conditions [9], TBC and boundaries of Han and Wu method and other artificial boundary conditions. Discrete TBCs yield more accurate results than others and conserve the stability of the scheme.

Düring analyses the numerical solution of the non-linear Black-Scholes equation [2]. He considers several compact finite difference schemes. These schemes were introduced by Rigal [11] and called R3-schemes. Also Düring investigates properties of these schemes. He found out the truncation errors.

In our thesis we apply different compact methods to American vanilla options. We apply different artificial boundary conditions for these scheme. We compare schemes with the same boundary conditions and initial data, also we compare each scheme with different boundary conditions. We will change the location of the left boundary to investigate its influence on the accuracy.

In section 2 we introduce the Black-Scholes equation. We apply the standard transformation to obtain the heat equation. We construct a finite domain and cover it by the grid. We choose the number of the space steps from the known parabolic mesh-ratio. Then we describe compact finite difference schemes: Method 1 is the method proposed by McCartin and Labadie [8]. Method 2 is the method described by Tangman et al. [13]. Methods R3A and R3B is proposed by Rigal in [11] and [2]. We distrain three types of boundary conditions: Mayfield, Han and Wu and the discrete TBC.

In section 3 we perform the numerical tests for the methods and different boundary conditions. Stability analysis and order of approximation are presented in this section.

In section 4 we summarize all results from section 3 and conclude how the different boundary conditions influence the properties of the schemes.

Chapter 2

Methods

Let us consider a financial market where risky assets (options, futures and forward contracts) can be traded. An option is an agreement between two sides: buyer and writer. The buyer has the right to exercise the option at the expiry date by the strike price. The simplest type of options is a vanilla option (Call or Put). Options are used to hedge the risk of portfolio and to speculate.

The European option has an exact expiry date as against the American option which can be exercised at any time before maturity. Therefore American options are more expensive and widely used on the markets.

2.1 The Black-Scholes equation

The Black-Scholes model is used to estimate the option value. To use this model we introduce the following notations:

Suppose that the process $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion. Therefore it satisfies the stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - D)dt + \sigma dW_t, \tag{2.1}$$

where $\{W_t\}_{t\geq 0}$ is the Brownian motion.

For European option the Black-Scholes equation has the following form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.2}$$

Conditions for European Call option:

· Terminal condition: $C(S,T) = max(S - E, 0)$;

Symbol	*Meaning*
S	Asset price
t	Time
V(S,t)	Value of option
C(S,t)	Value of Call option
P(S,t)	Value of Put option
E	Exercise price
T	Expiry date
r	Interest rate
σ	Volatility
D	Continuous dividend yield
S_f	Optimal exercise price

Figure 2.1: *Notations.*

· Boundary conditions: $C(S=0,t)=0, \quad t<T;$
$C(S\to\infty,t)\to Se^{-D(T-t)}.$

Conditions for European Put option:

· Terminal condition: $P(S,T)=max(E-S,0);$

· Boundary conditions: $P(S=0,t)=Ee^{-r(T-t)}, \quad t<T;$
$P(S\to\infty,t)\to 0.$

In the case of American options we have the possibility of early exercise. $S_f(t)$ is the switching point: if $0<S<S_f(t)$ then the option should be exercised; if $S>S_f(t)$ then the optimal strategy is to hold the option (in this case the American option is transformed to a European option).

Therefore the free boundary problem for an American option is the following:

$$I. \begin{cases} 0\le S<S_f(t), \quad \text{early exercise is optimal} \\ P=E-S, \\ \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D)S\frac{\partial V}{\partial S}-rV<0, \end{cases}$$

$$II. \begin{cases} S_f(t)\le S<\infty, \quad \text{early exercise is not optimal} \\ P>E-S, \\ \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D)S\frac{\partial V}{\partial S}-rV=0. \end{cases}$$

Conditions for American Call option:

· Boundary condition: $C(S_f(t),t)=max(S_f(t)-E,0);$

· Additional condition: $\frac{\partial C}{\partial S}(S_f(t),t)=1.$

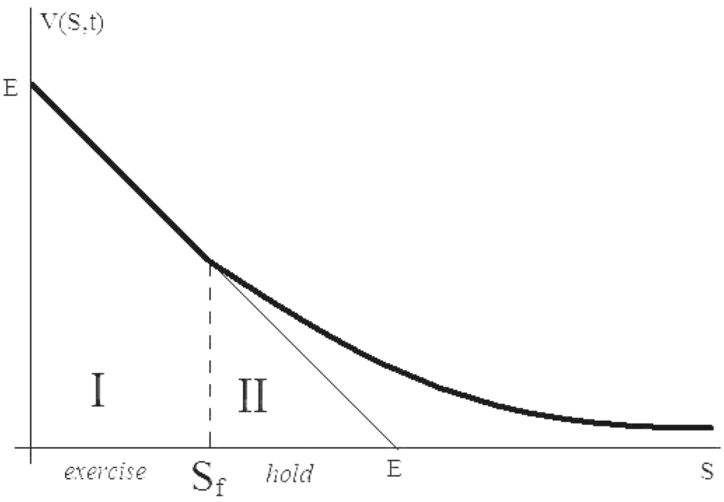

Figure 2.2: *American Put option*

The boundary conditions for American Put option.
· Boundary condition: $P(S_f(t), t) = max(E - S_f(t), 0)$;
· Additional condition: $\frac{\partial P}{\partial S}(S_f(t), t) = -1$.

2.2 The transformation of the Black-Scholes equation to the heat equation

The following standard change of variables can be used to transform the Black-Scholes equation to heat equation [14]

$$S = Ee^x, \quad t = T - 2\tau/\sigma^2, \quad V = E\upsilon(x, \tau). \tag{2.3}$$

After this substitution the Black-Scholes equation has the following form

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial x^2} + (k_2 - 1)\frac{\partial \upsilon}{\partial x} - k_1 \upsilon.$$

In this formula we set $k_1 = \frac{2r}{\sigma^2}$ and $k_2 = \frac{2(r-D)}{\sigma^2}$.

Now we can make one additional substitution of the dependent variables

$$\upsilon(x, \tau) = \exp[-\frac{1}{2}(k_2 - 1)x - \frac{1}{4}((k_2 - 1)^2 + 4k_1)\tau]u(x, \tau). \tag{2.4}$$

Finally, the Black-Scholes equation (2.2) takes the form of the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{2.5}$$

where

$$\tau \in \left[0, \frac{T\sigma^2}{2}\right],$$

and

$$x \in (-\infty, x_f(\tau)).$$

The next formula can be used to obtain the value of the option $V(S,t)$, which we are looking for

$$V(S,t) = E^{\frac{(k_2+1)}{2}} S^{\frac{1-k_2}{2}} \exp(-\frac{\sigma^2}{8}((k_2-1)^2 + 4k_1)(T-t))\times$$
$$u\left(\ln\frac{S}{E}, \frac{\sigma^2(T-t)}{2}\right). \tag{2.6}$$

For the European and American options the conditions will be changed under this transformation and become the following form [14].

Conditions after the transformation (2.6) for thr European Call option:

· the initial condition

$$u(x,0) = u_0(x) = e^{\frac{1}{2}(k_2-1)x} \max(e^x - 1, 0);$$

· the boundary condition

$$\lim_{x\to-\infty} u(x,\tau) = 0;$$

· the far-field condition

$$\lim_{x\to\infty} u(x,\tau) = \exp(\frac{1}{2}(k_2+1)x + \frac{1}{4}(k_2+1)^2\tau) - \exp(\frac{1}{2}(k_2-1)x + \frac{1}{4}(k_2-1)^2\tau).$$

The boundary and terminal conditions after transformation for the European Put option:

· the initial condition

$$u(x,0) = u_0(x) = e^{\frac{1}{2}(k_2-1)x} \max(1 - e^x, 0);$$

· the near-field condition

$$\lim_{x\to-\infty} u(x,\tau) = \exp(\frac{1}{2}(k_2-1)x + \frac{1}{4}(k_2-1)^2\tau) - \exp(\frac{1}{2}(k_2+1)x + \frac{1}{4}(k_2+1)^2\tau);$$

· the far-field condition

$$\lim_{x\to\infty} u(x,\tau) = 0.$$

For the American options this transformation modify the free boundary problem to the following parabolic variational inequality [14].

For the American Call option

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) \geq 0; \tag{2.7}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0; \tag{2.8}$$

$$g(x,\tau) = e^{\frac{1}{4}((k_2-1)^2+4k_1)\tau} \cdot \max(e^{\frac{1}{2}(k_2+1)x} - e^{\frac{1}{2}(k_2-1)x}, 0); \tag{2.9}$$

$$u(x,0) = u_0(x) = g(x,0); \tag{2.10}$$

$$\lim_{x\to\pm\infty} u(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau). \tag{2.11}$$

For the American Put option

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) \geq 0;$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0,$$

$$g(x,\tau) = e^{\frac{1}{4}((k_2-1)^2+4k_1)\tau} \cdot \max(e^{\frac{1}{2}(k_2-1)x} - e^{\frac{1}{2}(k_2+1)x}, 0),$$

$$u(x,0) = u_0(x) = g(x,0);$$

$$\lim_{x\to\pm\infty} u(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau).$$

2.3 The grid construction

For a numerical solution of the Black-Scholes equation we need to discretise the problem on a bounded computational domain $a < x < b$. For our calculation we use a uniform grid.

Let

$$\Omega_\tau = \{\tau_n \in \mathbb{R}_+ : \tau_n = n\Delta\tau, n = 0,1,...,N,\ \Delta\tau = \frac{T}{\sigma^2 N}\},$$

$$\Omega_x = \{x_j \in \mathbb{R} : x_j = a + j\Delta x, j = 0,1,...,M,\ \Delta x = \frac{b-a}{M}\},$$

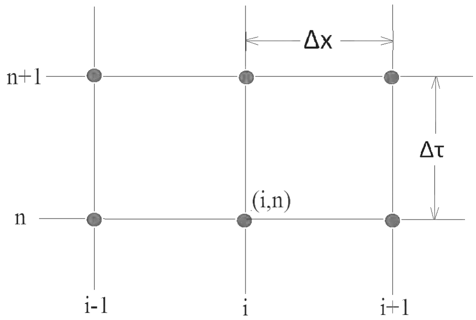

Figure 2.3: *A uniform 2D grid,* Δx *is the space step,* $\Delta\tau$ *is the time step.*

be a discrete computational domain with boundaries $a < 0$ and $b > 0$ in space and with boundaries $t = 0$ and $t = \frac{T}{\sigma^2}$ in time. We choose a and b to degrade the approximation error, see Figure 2.3.

Also we introduce a $(M-1)\times(N-1)$ matrix U of unknown values of the function $u(x,\tau)$ at the interior grid points, i.e. $u_j^n = u(x_j,\tau_n)$, $j = 1,..,M-1$, $n = 1,..,N-1$. To calculate the values of the function $u(x,\tau)$ at the boundary points we use the initial and boundary conditions, which will be described below.

The compact methods use only a small amount of points on one time level. We consider in the sequel two-level three-point schemes.

2.4 The description of the method 1

Here we describe a fast numerical method for the Black-Scholes equation for American options proposed by Han and Wu [5].

On the a-priori unknown free boundary of the early exercise $S_f(t)$ the solution of the Black-Scholes equation for an American Call option (2.2) should satisfy the following boundary conditions

$$C(S_f(t),t) = h(S_f(t)), \quad \frac{\partial C}{\partial S}(S_f(t),t) = 1, \quad 0 \le t \le T,$$

where $h(S) = \max(S-E,0)$.

After the standard transformation (2.3) for the Black-Scholes equation we obtain the dimensionless heat equation with the boundary and initial

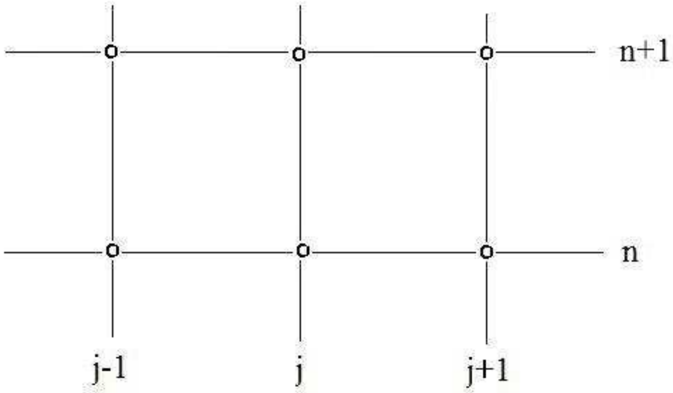

Figure 2.4: *The stencil for a high order compact scheme, j is numerate the space steps and n is the time levels.*

conditions (2.7) - (2.11) instead of the free boundary value problem. The boundary conditions are changed to the following form:

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau),$$

$$\alpha u(x_f(\tau), \tau) + \frac{\partial u(x_f(\tau), \tau)}{\partial x} = e^{(1-\alpha)x_f(\tau)-\beta\tau}, \quad 0 < \tau \leq \frac{\sigma^2 T}{2},$$

where the free boundary $x_f(\tau)$ is strictly positive for $0 < \tau \leq \frac{\sigma^2 T}{2}$ and parameters α and β have the form

$$\alpha = -\frac{\frac{2}{\sigma^2}(r-D) - 1}{2};$$

$$\beta = -\alpha^2 - \frac{2r}{\sigma^2}.$$

We consider the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x \leq x_f(\tau), \tag{2.12}$$

$$u(x, 0) = g(x, 0), \quad -\infty < x \leq x_f(0), \tag{2.13}$$

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau), \tag{2.14}$$

$$\alpha u(x_f(\tau),\tau) + \frac{\partial u(x_f(\tau),\tau)}{\partial x} = e^{(1-\alpha)x_f(\tau)-\beta\tau}, \quad 0 \le \tau \le \frac{\sigma^2 T}{2}, \tag{2.15}$$

$$u(x,\tau) \to 0, \quad x \to -\infty \tag{2.16}$$

on the a-priory unknown unbounded domain $\overline{\Omega}$. This domain can be presented as follows

$$\overline{\Omega} = \{(x,\tau)| -\infty < x < x_f(\tau), 0 < \tau \le \frac{\sigma^2 T}{2}\}.$$

We introduce an artificial boundary Γ_a at $x = a$

$$\Gamma_a = \{(x,\tau)|x = a, 0 < \tau \le \frac{\sigma^2 T}{2}\}.$$

This boundary Γ_a separates the region $\overline{\Omega}$ into two different areas: a bounded ('interior') domain Ω_i and an unbounded ('exterior') domain Ω_e, which can be written in the following form

$$\Omega_i = \{(x,\tau)|a < x < x_f(\tau), 0 < \tau \le \frac{\sigma^2 T}{2}\},$$

$$\Omega_e = \{(x,\tau)| -\infty < x < a, 0 < \tau \le \frac{\sigma^2 T}{2}\}.$$

We reduce the problem (2.12) - (2.16) on the bounded area Ω_i. It means that we have to find the corresponding boundary conditions on the artificial boundary Γ_a. The solution $u(x,\tau)$ for the problem (2.12) - (2.16) on the unbounded area Ω_e satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < a, \quad 0 < \tau \le \frac{\sigma^2 T}{2}, \tag{2.17}$$

$$u(x,0) = 0, \quad -\infty < x < a. \tag{2.18}$$

If we know the value $u(x,\tau)$ on the artificial boundary Γ_a

$$u(a,\tau) = \phi(\tau), \tag{2.19}$$

where $\phi(0) = 0$, then the problem (2.17) - (2.19) has the following solution [14]

$$u(x,\tau) = -\frac{(x-a)}{\sqrt{\pi}} \int_0^\tau e^{-\frac{(x-a)^2}{4(\tau-\lambda)}} \frac{\phi(\lambda)d\lambda}{(\tau-\lambda)^{3/2}}. \tag{2.20}$$

In formula (2.20) the following substitution has been made

$$\mu = \frac{(x-a)}{2\sqrt{\tau-\lambda}},$$

then it can be rewritten

$$u(x,\tau) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{(x-a)}{2\sqrt{\tau}}} e^{-\mu^2} \phi(\tau - \frac{(x-a)^2}{4\mu^2}) d\mu.$$

After this we can find the partial derivatives

$$\frac{\partial u(x,\tau)}{\partial x} = \frac{2}{\sqrt{\pi}} \phi(0) e^{-\frac{(x-a)^2}{4\tau}} \frac{1}{2\sqrt{\tau}}$$

$$+\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{(x-a)}{2\sqrt{\tau}}} \phi'(\tau - \frac{(x-a)^2}{4\mu^2})(-\frac{x-a}{2\mu^2}) e^{-\mu^2} d\mu$$

$$= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{(x-a)}{2\sqrt{\tau}}} \phi'(\tau - \frac{(x-a)^2}{4\mu^2})(\frac{x-a}{2\mu^2}) e^{-\mu^2} d\mu.$$

It means

$$\frac{\partial u(x,\tau)}{\partial x} = \frac{1}{\sqrt{\pi}} \int_0^{\tau} e^{-\frac{(x-a)^2}{4(\tau-\lambda)}} \frac{\phi'(\lambda) d\lambda}{\sqrt{\tau-\lambda}}.$$

The value of this partial derivative in point $x = a$

$$\frac{\partial u}{\partial x}|_{x=a} = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\partial u(a,\lambda)}{\partial \lambda} \frac{d\lambda}{\sqrt{t-\lambda}}. \tag{2.21}$$

The formula (2.21) is the exact boundary condition for the problem (2.12) - (2.16) on the artificial boundary Γ_a. By the relation (2.21) we have a bounded problem on area Ω_i for the American Call option against the unbounded problem on the area $\overline{\Omega}$ for the same American Call option [14]

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad a < x \le x_f(\tau), \quad 0 \le \tau \le \frac{\sigma^2 T}{2}, \tag{2.22}$$

$$u(x,0) = g(x,0), \quad a < x \le x_f(0), \tag{2.23}$$

$$u(x_f(\tau),\tau) = g(x_f(\tau),\tau), \tag{2.24}$$

$$e^{(1-\alpha)x_f(\tau)-\beta\tau} \left[\alpha u(x_f(\tau),\tau) + \frac{\partial u(x_f(\tau),\tau)}{\partial x} \right] = 1, \quad 0 \le \tau \le \frac{\sigma^2 T}{2}, \tag{2.25}$$

$$\frac{\partial u}{\partial x}|_{x=a} = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\partial u(a,\lambda)}{\partial \lambda} \frac{d\lambda}{\sqrt{t-\lambda}}. \tag{2.26}$$

the finite difference approximation.

This subsection is dedicated to the numerical approximation of the above mentioned problem (2.22) - (2.26). In the first step we have to define the corresponding two dimensional grid. For this let's introduce some notations.

We denote $\gamma = \frac{\triangle\tau}{(\triangle x)^2}$. The parameter γ is called the parabolic mesh ratio and plays a central role in the stability and accuracy considerations of the heat equation.

Equation (2.22) is the heat equation on the bounded area Ω_i. We want to find the numerical solution of this problem. It can be solved by using the different finite difference methods. In this thesis we will consider the Crandall-Douglas scheme for the heat equation. This scheme is an accurate numerical procedure and is efficient since it used only six points on two time levels.

The discretization by the Crandall-Douglas scheme of the diffusion operator has the following form:

$$\begin{aligned} &(1-6\gamma)u_{j+1}^n + (10+12\gamma)u_j^n + (1-6\gamma)u_{j-1}^n \\ &= (1+6\gamma)u_{j+1}^{n-1} + (10-12\gamma)u_j^{n-1} + (1+6\gamma)u_{j-1}^{n-1}, \end{aligned} \tag{2.27}$$

$j = 1, 2, ..., \quad n = 1, 2, ..$

For $j = 1$ this formula can be rewritten as

$$(1-6\gamma)u_2^n + (10+12\gamma)u_1^n = b_1,$$

where

$$b_1 = (1+6\gamma)(u_2^{n-1} + u_0^{n-1}) + (10-12\gamma)u_1^{n-1} - (1-6\gamma)u_0^n. \tag{2.28}$$

For other value $j = 2, 3, ...$

$$(1-6\gamma)u_{j+1}^n + (10+12\gamma)u_j^n + (1-6\gamma)u_{j-1}^n = b_j,$$

where

$$b_j = (1+6\gamma)(u_{j+1}^{n-1} + u_{j-1}^{n-1}) + (10-12\gamma)u_j^{n-1}.$$

The matrix form of this method is $Mu = b$, where

$$M = \begin{pmatrix} (10+12\gamma) & (1-6\gamma) & 0 & \dots & 0 \\ (1-6\gamma) & (10+12\gamma) & (1-6\gamma) & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & (1-6\gamma) & (10+12\gamma) \end{pmatrix},$$

$$u = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \end{pmatrix}.$$

We rewrite the Crandall-Douglas scheme as a θ-method. In the general form the θ-method can be presented as follows

$$-\theta\gamma u_{j-1}^{n+1} + (1+2\theta\gamma)u_j^{n+1} - \theta\gamma u_{j+1}^{n+1}$$
$$= (1-\theta)\gamma u_{j-1}^n + (1-2(1-\theta)\gamma)u_j^n + (1-\theta)\gamma u_{j+1}^n.$$

If we divide each equation in the Crandall-Douglas scheme by 12 we obtain

$$\left(\frac{1}{12} - \frac{1}{2}\gamma\right) u_{j+1}^n + \left(\frac{10}{12} + \gamma\right) u_j^n + \left(\frac{1}{12} - \frac{1}{2}\gamma\right) u_{j-1}^n$$
$$= \left(\frac{1}{12} + \frac{1}{2}\gamma\right) u_{j+1}^{n-1} + \left(\frac{10}{12} - \gamma\right) u_j^{n-1} + \left(\frac{1}{12} + \frac{1}{2}\gamma\right) u_{j-1}^{n-1}$$

Now we can find θ:

$$-\theta\gamma = \left(\frac{1}{12} - \frac{1}{2}\gamma\right),$$
$$\theta = \frac{1}{2} - \frac{1}{12\gamma}.$$

For the control of the correct calculations we can put θ in the general form for θ-methods. Then we obtain

$$-\left(\frac{1}{2} - \frac{1}{12\gamma}\right)\gamma u_{j-1}^{n+1} + \left(\gamma + \frac{5}{6}\right) u_j^{n+1} - \left(\frac{1}{2} - \frac{1}{12\gamma}\right) u_{j+1}^{n+1} \tag{2.29}$$
$$= \frac{1}{2} + \frac{1}{12\gamma}\gamma u_{j-1}^n + \left(\frac{5}{6} - \gamma\right) u_j^n + \left(\frac{1}{2} + \frac{1}{12\gamma}\right)\gamma u_{j+1}^n.$$

This scheme in a facsimile is the Crandall-Douglas scheme.

We introduce a new notation of variables

$$s_1 = (10 + 12\gamma), \quad y_1 = b_1.$$

For $j = 1$

$$s_1 u_1^n + (1-6\gamma)u_2^n = y_1 \quad \Rightarrow u_1^n = \frac{b_1 - (1-6\gamma)u_2^n}{s_1}.$$

For $j = 2$ this formula has the following form

$$(1-6\gamma)u_3^n + (10+12\gamma)u_2^n + (1-6\gamma)u_1^n = y_2,$$

$$(1-6\gamma)u_3^n + ((10+12\gamma) - \frac{(1-6\gamma)^2}{s_1})u_2^n = b_2 - \frac{y_1(1-6\gamma)}{s_1}.$$

In general case

$$s_j u_j^n + (1-6\gamma)u_{j+1}^n = y_j,$$

where

$$s_j = (10+12\gamma) - \frac{(1-6\gamma)^2}{s_{j-1}},$$

and

$$y_j = b_j - \frac{(1-6\gamma)y_{j-1}}{s_{j-1}}.$$

We have an inequality $u_{M_\varepsilon}^n \geq g_{M_\varepsilon}$.

Assume that M_ε is the biggest number which satisfies this inequality, then we have

$$u_{M_\varepsilon}^n = \frac{1}{s_{M_\varepsilon}} \left(b_{M_\varepsilon} - \frac{(1-6\gamma)y_{M_\varepsilon - 1}}{s_{M_\varepsilon - 1}} - (1-6\gamma)g_{M_\varepsilon + 1}, \right),$$

$$u_j^n = \frac{1}{s_j}(y_j - (1-6\gamma)u_{j+1}^n),$$

for $j = M_\varepsilon - 1, M_\varepsilon - 2, ...$

2.5 The description of the method 2

We describe the method proposed by Tangman, Gopaul and Bhuruth [13] and call it Method 2. This method is an improvement of the Hang and Wu method [5]. Authors consider the difference between American and European options. Price of both types of options is the solution of the Black-Scholes equation. So, this difference also satisfy the Black-Scholes equation. We use the standard transformations [14] to obtain the heat equation

$$\frac{\partial u_D}{\partial \tau} = \frac{\partial^2 u_D}{\partial x^2}, \; x_f(\tau) \leq x < \infty,$$

with the boundary and initial conditions

$$u_D(x,0) = 0, \; x_f(0) \leq x < \infty,$$

$$u_D(x_f(\tau),\tau) = h(x_f(\tau),\tau) - u_E(x_f(\tau),\tau), \; 0 \leq \tau \leq \tau_{max},$$

$$u_D(x,\tau) \to 0 \; as \; x \to \infty,$$

where u_E is the transformed function of value for an European put option, u_D denotes the difference between American and European options.

We begin with applying this scheme for the American option instead of the difference u_D. Then we apply it to the difference u_D as noted above.

We just change the initial conditions. In the case of the non-uniform grid at the n-th time level the finite-difference approximation of the second space derivative $\frac{\partial^2 u}{\partial x^2}$ and the function $u(x,\tau)$ are proposed in [12] to achieve the maximal accuracy

$$D_{xx}[u_j^n] = \frac{2}{x_{j+1}-x_{j-1}}\left(\frac{u_{j+1}^n - u_j^n}{x_{j+1}-x_j} - \frac{u_j^n - u_{j-1}^n}{x_j - x_{j-1}}\right), \tag{2.30}$$

$$D[u_j^n] = \frac{1}{3}(u_{j-1}^n + u_j^n + u_{j+1}^n) - \frac{b_j}{3}D_{xx}[u^n], \tag{2.31}$$

where

$$b_j = \frac{1}{3}\left((x_j - x_{j-1})^2 + (x_j - x_{j-1})(x_{j+1}-x_j) + (x_{j+1}-x_j)^2\right)$$

is an effective mean square grid spacing.

Instead of the common forward approximation of the first derivative Tangman et al. [13] consider the forward difference between approximations of the function (2.31) on the n-th and $(n+1)$-th time level. Then we can introduce the weighted scheme

$$\frac{D[u_j^{n+1}] - D[u_j^n]}{\Delta\tau} = (\frac{1}{2}\lambda - \widetilde{\gamma})D_{xx}[u_j^{n+1}] + (\frac{1}{2}\lambda + \widetilde{\gamma})D_{xx}[u_j^n]. \tag{2.32}$$

To obtain the fourth order of an approximation we choose optimal values of the parameters $\lambda = 1$ and $\widetilde{\gamma} = \frac{b_j}{12\Delta\tau}$. For the case of an uniform grid it quotes to the optimal weighted scheme.

The replacement of (2.30) and (2.30) into (2.32) introduces the high order compact (HOC) coefficients $\alpha_{-1}(j)$, $\alpha_0(j)$ and $\alpha_1(j)$, which are calculated at each time step. This is the set of coefficients from the scheme

$$\alpha_{-1}(j) = \frac{1}{3\Delta\tau}\left(1 - \frac{2b_j}{(x_{j+1}-x_{j-1})(x_j - x_{j-1})}\right) \tag{2.33}$$

$$-(\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2}{(x_{j+1}-x_{j-1})(x_j - x_{j-1})},$$

$$\alpha_0(j) = \frac{1}{3\Delta\tau}(1 + \frac{2b_j}{(x_{j+1}-x_{j-1})}\left(\frac{1}{x_{j+1}-x_j} + \frac{1}{x_j - x_{j-1}})\right) \tag{2.34}$$

$$+(\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2}{x_{j+1}-x_{j-1}}\left(\frac{1}{x_{j+1}-x_j} + \frac{1}{x_j - x_{j-1}}\right),$$

$$\alpha_1(j) = \frac{1}{3\Delta\tau}\left(1 - \frac{2b_j}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}\right) \tag{2.35}$$
$$-(\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)},$$

where $b_j = \frac{1}{3}\left((x_j - x_{j-1})^2 + (x_j - x_{j-1})(x_{j+1}-x_j) + (x_{j+1}-x_j)^2\right)$.
Now we can present the scheme (2.32) as a tridiagonal linear system

$$\begin{pmatrix} \alpha_0(1) & \alpha_1(1) & 0 & ... & \\ \alpha_{-1}(2) & \alpha_0(2) & \alpha_1(2) & 0 & ... \\ \ddots & & & & \\ 0 & ... & \alpha_{-1}(m-2) & \alpha_0(m-2) & \alpha_1(m-2) \\ 0 & ... & 0 & \alpha_{-1}(m-1) & \alpha_0(m-1) \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-2}^{n+1} \\ u_{m-1}^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_{m-2} \\ \hat{b}_{m-1} \end{pmatrix}.$$

But in the case of the uniform grid the coefficients reduce to

$$b_j = \Delta x^2.$$

If we put this expression into (2.33) - (2.35) and multiply by $\Delta\tau$, we obtain

$$\alpha_{-1}(j) = \frac{1}{3}\left(1 - \frac{2\Delta x^2}{2\Delta x^2}\right) - (\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2\Delta\tau}{2\Delta x^2} = -(\frac{1}{2}\lambda - \widetilde{\gamma})\gamma,$$
$$\alpha_0(j) = \frac{1}{3}(1 + \frac{2\Delta x^2}{2\Delta x^2}) + (\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2\Delta\tau}{2\Delta x^2} = 1 + (\lambda - 2\widetilde{\gamma})\gamma,$$
$$\alpha_1(j) = \frac{1}{3}\left(1 - \frac{2\Delta x^2}{2\Delta x^2}\right) - (\frac{1}{2}\lambda - \widetilde{\gamma})\frac{2\Delta\tau}{2\Delta x^2} = -(\frac{1}{2}\lambda + \widetilde{\gamma})\gamma.$$

As we can see, there is no dependence on the spatial index j, i.e. the coefficients in the scheme α_{-1}, α_0, α_1 are constant with $\alpha_{-1} = \alpha_1$. We achieve the tridiagonal matrix with constant coefficients.

Similarly we find the vector on the right-hand-side $\hat{b}_j$. In the case of the uniform grid coefficients before u_{j-1}^n, u_j^n, u_{j+1}^n respectively, are

$$\hat{\alpha}_{-1} = (\frac{1}{2}\lambda + \widetilde{\gamma})\gamma,$$

$$\hat{\alpha}_0 = 1 + (\lambda + 2\widetilde{\gamma})\gamma,$$

$$\hat{\alpha}_1 = (\frac{1}{2}\lambda + \widetilde{\gamma})\gamma.$$

Then

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_{m-2} \\ \hat{b}_{m-1} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_0 & \hat{\alpha}_1 & 0 & ... & \\ \hat{\alpha}_{-1} & \hat{\alpha}_0 & \hat{\alpha}_1 & 0 & ... \\ \ddots & & & & \\ 0 & ... & \hat{\alpha}_{-1} & \hat{\alpha}_0 & \hat{\alpha}_1 \\ 0 & ... & 0 & \hat{\alpha}_{-1} & \hat{\alpha}_0 \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{m-2}^n \\ u_{m-1}^n \end{pmatrix}$$

In this thesis we investigate the three types of left boundary conditions. On the right free boundary we use a Dirichlet condition. We describe them below.

To clarify this method we consider the scheme as the θ-method (or a weighted average method). In the general case this scheme has the following form

$$\begin{aligned} &-\theta\gamma u_{j-1}^{n+1} + (1+2\theta\gamma)u_j^{n+1} - \theta\gamma u_{j+1}^{n+1} \\ &= (1-\theta)\gamma u_{j-1}^n + (1-2(1-\theta)\gamma)u_j^n + (1-\theta)\gamma u_{j+1}^n. \end{aligned} \tag{2.36}$$

Or in compact form it can be presented as

$$-\theta D_{xx}u_j^{n+1} + \frac{u_j^{n+1}}{\Delta\tau} = \frac{u_j^n}{\Delta\tau} + (1-\theta)D_{xx}u_j^n, \tag{2.37}$$

where $D_{xx}u_j^n$ is the second order standard difference quotient.

If we put $\theta = \frac{1}{2} - \frac{(\Delta x)^2}{12\Delta\tau} = \frac{1}{2} - \frac{1}{12\gamma}$, then Method 2 (2.32) is transformed to the θ-method (2.36) and the matrix A on the left-hand-side has the diagonal entries $(1+2\theta\gamma)$ and the off-diagonal entries $-\theta\gamma$. The matrix B at the right-hand side has the entries $(1-2(1-\theta)\gamma)$ and the off-diagonal entries $(1-\theta)\gamma$.

The θ-methods are a very wide group of methods of numerical approximations to the heat equation. We can obtain different schemes by choice of the implicitness parameter θ. If we take $\theta = 0.5$ we obtain the well-known Crank-Nicolson method. For $\theta = 1$ the method is called implicit, vice versa, if $\theta = 0$ it is a fully explicit method. Thus θ is the measure of implicity of the scheme. In the Method 1 we have $\theta = 12 \cdot (\frac{1}{2} - \frac{1}{12\gamma})$, so if we divide the scheme equation (2.27) over 12, we obtain the scheme (2.29) with

$$\theta = \frac{1}{2} - \frac{1}{12\gamma}. \tag{2.38}$$

This θ is called *the optimal weighting parameter*. The same value of θ is considered in the Method 1. Hence, that considered two Methods coincide for the heat equation. However, for other possibly nonlinear equations these methods can give different results.

2.6 The description of R3-Methods

Another group of methods was proposed by Rigal [11] and then applied to the Black-Scholes equation by Düring in his dissertation [2]. These schemes have the fourth order in space and the second order in time. The methods are based on a general two-level three-point scheme

$$(1+C)D_t u_j^n = (\frac{1}{2}+A_1)D_{xx}u_j^n + (\frac{1}{2}+A_2)D_{xx}u_j^{n+1}$$
$$-f(\frac{1}{2}+B_1)D_x u_j^n - f(\frac{1}{2}+B_2)D_x u_j^{n+1}, \quad (2.39)$$

where A_j, B_j and C are real numbers chosen to eliminate the lower order terms in the truncation error [11]. But in (2.5) there are no first derivatives, i.e. $f = 0$. The coefficients A, B and C are chosen to make the error of approximation nearby zero. The choice of C influences just weakly, so Rigal considered $C = 0$ and express A_1, A_2 and B_1 from the scheme as a function of B_2. Also there are presented theorems about coefficients in the scheme.

According to the Theorem 1 in [11], "*A necessary condition of consistency of the scheme is* $C = B_1 + B_2$". This condition stipulates the relation between B_1 and B_2

$$B_1 = -B_2.$$

To achieve the high order of an approximation we eliminate terms of the order lower than 2 in time and 4 in space from the truncation error [11]. We chose parameters A and B by this condition and obtain

$$A_2 = -\frac{f^2\Delta\tau}{12} - \frac{1}{6\gamma} - B_2\left(1+\frac{f^2\Delta\tau}{2}\right),$$
$$A_1 = \frac{f^2\Delta\tau}{12} + \frac{1}{6\gamma} + B_2\left(1-\frac{f^2\Delta\tau}{2}\right).$$

If we take these values of parameters A and B, the scheme has the second order in time and the fourth order in space. For the case $f = 0$ these constraints can be rewritten in the following form

$$B_1 = -B_2, \quad (2.40)$$
$$A_1 = \frac{1}{6\gamma} + B_2, \quad (2.41)$$
$$A_2 = -\frac{1}{6\gamma} - B_2. \quad (2.42)$$

Now we obtain different schemes depending on the choice of B_2.

The R3A-method

For the R3A-method Rigal offers

$$B_2 = -\frac{1}{12\gamma}. \tag{2.43}$$

Then we find residuary constants from (2.40) -(2.42)

$$B_1 = \frac{1}{12\gamma},$$

$$A_1 = \frac{1}{6\gamma} - \frac{1}{12\gamma} = \frac{1}{12\gamma},$$

$$A_2 = -\frac{1}{6\gamma} + \frac{1}{12\gamma} = -\frac{1}{12\gamma}.$$

And if we put these coefficients to the (2.39), we obtain the scheme

$$D_t u_j^n = (\frac{1}{2} + \frac{1}{12\gamma})D_{xx}u_j^n + (\frac{1}{2} - \frac{1}{12\gamma})D_{xx}u_j^{n+1}.$$

Also we can obtain coefficients of the tridiagonal matrices A and B, which describe the difference scheme, given by Rigal [11]

$$a_0 = \frac{5}{6} + \gamma,$$

$$a_1 = \frac{1}{12} - \frac{\gamma}{2},$$

$$a_{-1} = \frac{1}{12} - \frac{\gamma}{2},$$

$$b_0 = \frac{5}{6} - \gamma,$$

$$b_1 = \frac{1}{12} + \frac{\gamma}{2},$$

$$b_{-1} = \frac{1}{12} + \frac{\gamma}{2}.$$

Obviously, we obtain again the optimal weighted scheme for the heat equation.

The R3B-method

If we take another value of B_2 than in (2.43), we obtain another scheme. For the R3B-method Rigal take

$$B_2 = -\frac{1}{12\gamma} - \frac{f^2\Delta\tau}{12}.$$

But we have $f = 0$, it means that we obtain the same B_2, and, consequently, we have the same coefficients as in the R3A-method.

2.7 The description of the non-compact Method

The considered schemes so far are the high order compact schemes. It means that a numerical solution of the heat equation is obtained by using a small number of points, but the solution has the high order of accuracy. There exist another way to achieve an accurate solution. For example, by using more points in the stencil. We consider the fully explicit scheme with five points on n-th time level. The only unknown variable u_j^{n+1} is in the left part of the algebraic formula below that was derived from the discretization.

The space derivative is approximated on n-th time level, which is known. This method is an explicit modification of the Crank-Nicolson scheme

$$\frac{u^{n+1} - u^n}{\Delta\tau} = \frac{1}{2}D_{xx}u^n + \frac{1}{2}D_{xx}u^{n+1}. \quad (2.44)$$

Then from the explicit Euler scheme we obtain an aproximation for u^{n+1} on the right hand side of (2.44)

$$u^{n+1} = u^n + \Delta\tau D_{xx}u^n.$$

If we insert this expression into (2.44) then we obtain the explicit expression for the unknown value of the function on the next time level u_j^{n+1}

$$u_j^{n+1} = u_j^n + \frac{\Delta\tau}{2}(D_{xx})u_j^n + \frac{\Delta\tau}{2}D_{xx}(u_j^n + \Delta\tau D_{xx}u_j^n), \quad (2.45)$$

$$u_j^{n+1} = u_j^n + \frac{\gamma}{2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) + \frac{\gamma}{2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n$$

$$+\gamma(u_{j-2}^n - 2u_{j-1}^n + u_j^n - 2\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + u_j^n - 2u_{j+1}^n + u_{j+2}^n). \quad (2.46)$$

This method is called the Heun's method. The main properties of this scheme are considered in chapter 3.

After some transformation we obtain the fully explicit scheme of the two-level five-point scheme. It means that for numerical solution we use two time

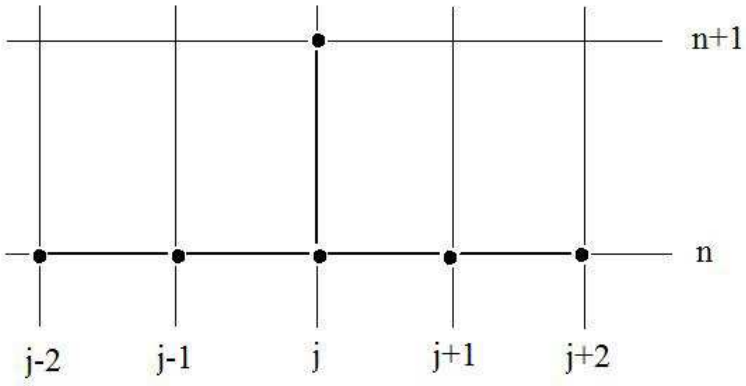

Figure 2.5: *The non-Compact explicit 2-level 5-point Scheme,j is numerate the space steps and n is the time levels.*

levels n and $n+1$. On the n-th level we use 5 points to find the value on the next time level. This scheme has the following form

$$u_j^{n+1} = u_j^n + \gamma(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

$$+\frac{\gamma^2}{2}(u_{j-2}^n - 4u_{j-1}^n + 6u_j^n - 4u_{j+1}^n + u_{j+2}^n) \qquad (2.47)$$

This is the 5-point scheme, so we have to use additional boundary conditions. We chose the Crank-Nicolson scheme for the second and for the $M-1$ points.

2.8 Artificial Boundary Conditions

To solve free boundary problem (2.12) - (2.16) numerically, many different methods are developed, e.g. the standard method consists in the reformulation to a linear complementary problem (LCP) and the solution by the projected SOR method of Cryer [1]. Alternatively, the penalty and the front-fixing methods were developed (e.g. in [4], [10]). A disadvantage of these methods is the change of the underlying model. A different approach [6] is based on a recursive calculation of the early exercise boundary, estimating the boundary only at some points and then approximating the whole boundary by the Richardson extrapolation. Explicit boundary tracking algorithms are e.g. a finite difference bisection scheme [7] or the *front–tracking strategy of Han and Wu* [5]. In this thesis we consider solely the later approach of Han and Wu, which we describe now briefly.

The artificial boundary conditions of Han and Wu.

The solution $u(x,\tau)$ of the problem (2.12) - (2.16) should satisfies the following condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\partial u(a,\lambda)}{\partial \lambda} \frac{d\lambda}{\sqrt{t-\lambda}}.$$

It can be approximated in this way

$$\frac{u_1^n - u_{-1}^n}{2\triangle x} = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{n} \frac{2(u_0^m - u_0^{m-1})}{\sqrt{\tau_n - \tau_m} + \sqrt{\tau_n - \tau_{m-1}}}. \tag{2.48}$$

An implicit approximation of the heat equation is given by

$$\frac{u_0^n - u_0^{n-1}}{\triangle \tau} = \frac{u_1^n - 2u_0^n + u_{-1}^n}{(\triangle x)^2}. \tag{2.49}$$

From equation (2.48) we express the value u_{-1}^n and put it in equation (2.49). After this replacement we obtain the boundary u_0^n

$$u_0^n = \frac{\sqrt{\gamma} H_1 + \sqrt{\pi} H_2/4}{\sqrt{\gamma} + \sqrt{\pi}(1+2\gamma)/4}, \tag{2.50}$$

where

$$H_1 = u_0^{n-1} + \sqrt{\pi}\gamma \frac{u_1^n}{4} - \sum_{m=1}^{n-1} \frac{(u_0^m - u_0^{m-1})}{\sqrt{n-m} + \sqrt{n-(m-1)}},$$
$$H_2 = u_0^{n-1} + \gamma u_1^n.$$

The other type of the boundary conditions was presented by Mayfield [9]. This condition is an ad-hoc discretization strategy, which can be applied to the heat equation.

The discretized TBC of Mayfield

Mayfield in her paper [9] considers the Schrödinger equation and finds boundary conditions on the left boundary. One way of the discretization the analytic transparent boundary conditions at $x = a$ is

$$u(a,\tau) = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\partial u(a,\lambda)}{\partial \lambda} \frac{d\lambda}{\sqrt{\tau - \lambda}}, \tag{2.51}$$

The approximation of the first integral has the following form

$$\int_0^{\tau_m} \frac{\partial u(a,\lambda)}{\partial \lambda} \frac{d\lambda}{\sqrt{\tau_m - \lambda}} = \frac{1}{\triangle x} \sum_{m=0}^{n-1} (u_1^{n-m} - u_0^{n-m}) \int_{\tau_m}^{\tau_{m+1}} \frac{d\xi}{\sqrt{\xi}}.$$

$$= \frac{2\sqrt{\triangle t}}{\triangle x} \sum_{m=0}^{n-1} \frac{(u_1^{n-m} - u_0^{n-m})}{\sqrt{m+1} + \sqrt{m}}$$

Let us denote the convolution coefficients as

$$\tilde{l}_m = \frac{1}{\sqrt{m+1} + \sqrt{m}}.$$

If we put the approximation of this integral into formula (2.51) then we obtain the discretized transparent boundary conditions of Mayfield

$$u_1^n - u_0^n = \frac{\sqrt{\pi} \triangle x}{2\sqrt{\triangle t}} u_0^n - \sum_{m=1}^{n-1} (u_1^{n-m} - u_0^{n-m}) \tilde{l}_m.$$

The discretized transparent boundary conditions of Mayfield have a new form now

$$u_1^n - \left(1 + \frac{\sqrt{\pi} \triangle x}{2\sqrt{\triangle t}}\right) u_0^n = \sum_{m=1}^{n-1} (u_1^{n-m} - u_0^{n-m}) \tilde{l}_m.$$

The discrete TBC

The third type of the boundary conditions we used for the schemes is the discrete TBC [3]. It does not involve additional computational costs, but it prevents any numerical reflections at the boundary. The discrete TBC doesn't destroy the stability of the underlying scheme for the interior points. These boundary conditions are used for the θ-methods and depend on this parameter.

We use the discrete TBC at the left boundary. So, the underlying equation for the discretization in [3] was the following

$$\Delta_\tau^+ u_j^n = \gamma \Delta_x^2 u_j^{n+\theta},$$

where

$$\Delta_\tau^+ = \Delta\tau \cdot D_\tau^+,$$

$$\Delta_x^2 = \Delta x^2 D_{xx},$$

for the equation $u^{n+\theta} = \theta u^{n+1} + (1-\theta)u^n$.

Now we construct the discrete TBC for the left boundary $x_0 = a$

$$u_1^n = u_0^n * l^{(n)} = \sum_{k=1}^{n} u_0^k l^{(n-k)},$$

where convolution coefficients are given by

$$l^{(0)} = 1 + \frac{1}{2\theta\gamma} + \frac{2\theta}{\gamma}\sqrt{A},$$
$$l^{(n)} = \frac{(-1)^n}{2\theta(1-\theta)\gamma}(\frac{1-\theta}{\theta})^n + \frac{1}{\gamma\sqrt{A}}\cdot$$
$$\left(\frac{A}{2\theta}\tilde{P}_n(\mu) - \frac{C}{2(1-\theta)}\tilde{P}_{n-1}(\mu) + \frac{1}{2\theta(1-\theta)^2}\sum_{k=0}^{n-1}\left(-\frac{1-\theta}{\theta}\right)^{n-k}\tilde{P}_k(\mu)\right),$$

where $\tilde{P}_n(\mu) = P_n(\mu)(\frac{\sqrt[+]{C}}{\sqrt{A}})^n$ is "damped" Legendre polynomials and the parameter $\mu = \frac{B}{\sqrt{A}\sqrt[+]{C}}$. The coefficients A, B, and C are given by

$$A = 1 + 4\gamma\theta,$$
$$B = 1 - 2\gamma(1-2\theta),$$
$$C = 1 - 4\gamma(1-\theta).$$

The right free boundary condition

There are 2 types of the boundary conditions. If the unknown variable on the boundary is a known function on time or a constant, it is called the Dirichlet BC:

$$u(x_f, \tau) = f(\tau). \tag{2.52}$$

Another type of boundary conditions is the Neumann BC, if we set the space derivative as a known function or as a constant

$$\frac{\partial u}{\partial x}(x_f, \tau) = g(\tau). \tag{2.53}$$

In the Dirichlet conditions we use the exact solution on the boundary. It means that the singular source of an error is the approximation of derivatives in the heat equation. But if we use the Neumann BC, we also approximate the first space derivative at the boundary, so we have an additional source of errors. The order of the one-side approximation is $\mathcal{O}(\Delta x)$, but for the interior points we use the fourth order compact scheme. This is a problem for the accuracy estimation, because the function behavior at the boundaries influence the numerical solution. We can use the central approximation. It has the second order of the approximation. We need one additional right point for that. This point does not exist in the computational domain.

Different types of the boundary condition influence on the accuracy of the solution. For example, the Crank-Nicolson scheme with the Dirichlet conditions is unconditionally stable. With Neumann BC it is also stable, but there appears the oscillation.

In our thesis we use the Dirichlet BC on the right free boundary. We use the fact, that on the free boundary the value of the option is equal to the payoff function, i.e. (2.14) is satisfied. But we should note, that the value $x_f(\tau)$ is not given. It's a part of a solution. The explicit definition of the free boundary is a difficult question. We want to determine the option value in the connection with the free boundary. It means that we have to determine the free boundary numerically. We use the Han and Wu front-tracking strategy for that [5].

Chapter 3

Results

3.1 The stability analysis

A theoretical analysis of the schemes (2.27), (2.32), (2.39) and (2.45) will be made in this section. The numerical solution of the heat equation is computed on the bounded grid. We should expect that more accurate solutions can be obtained on the refined grid. But the computational costs increase. Therefore sometimes it is more reasonable to use high order schemes on the coarse grid than low order schemes on the refined grid. These reflections lead to an idea of a computational efficiency. Important questions besides efficiency are stability and convergence of numerical schemes.

Definition (Stability). A scheme is stable if the difference between numerical and analytical solutions remains bounded as the number of time steps tend to infinity.

Definition (Convergence). A scheme is convergent if the difference between numerical and analytical solutions at a fixed point tends to zero uniformly if space and time steps tend to zero.

Definition (Error of approximation). The difference between an analytical solution of the partial differential equation and the numerical solution is called error of approximation. The value of the error in the (i, n) point depends on values of $\triangle x$, $\triangle \tau$ and the values of the high order derivatives, which were not included in the finite difference approximation of the derivatives in the considered differential equation.

Only a small family of the problems has a known analytical solution. If the problem has not it then a conclusion about the convergence can be made by the construction of numerical solution on a sequentially refining grid.

The von Neumann method is the most efficient when defining stability criteria. This method can be applied only for linear problems with initial conditions and constant coefficients. Loosely speaking, the von Neumann method is applied for interior points only.

The optimal weighed θ-method

We consider the θ-scheme for the heat equation

$$D_\tau^+ u_j^n = \theta D_x^2 u_j^{n+1} + (1-\theta) D_x^2 u_j^n,$$

where D_τ^+ is the forward difference and D_x^2 is the standard second order difference:

$$D_\tau^+ = \frac{u_j^{n+1} - u_j^n}{\triangle\tau},$$

$$D_x^2 = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2}.$$

Applying to the scheme the discrete separation of variables ansatz $u_j^n = v_j w^n$, we obtain

$$v_j \frac{w^{n+1} - w^n}{\triangle\tau} = \theta w^{n+1} D_x^2 v_j + (1-\theta) w_n D_x^2 v_j = [\theta w^{n+1} + (1-\theta) w^n] D_x^2 v_j$$

Solving with respect to temporal and spatial dependence yields

$$\frac{w^{n+1} - w^n}{\triangle\tau[\theta w^{n+1} + (1-\theta) w^n]} = \frac{D_x^2 v_j}{v_j} = -\delta = const. \tag{3.1}$$

We start with the right hand side of (3.1)

$$D_x^2 v_j + \delta v_j = 0, \quad j = 1, .., M-1,$$

or

$$v_{j+1} + (\delta(\triangle x)^2 - 2) v_j + v_{j-1} = 0, \quad j = 1, .., M-1. \tag{3.2}$$

Let $v_j^{(l)} = \sqrt{2} \sin(\pi l x_j), \quad j = 0, .., M, \quad l = 1, .., M-1.$
Put this expression to (3.2), note that $x_{j\pm1} = x_j \pm \triangle x$

$$\sin \pi l(x_j + \triangle x) + (\delta(\triangle x)^2 - 2) \sin(\pi l x_j) + \sin \pi l(x_j - \triangle x)$$

$$= [2\cos(\pi l \triangle x) + \delta(\triangle x)^2 - 2] \sin(\pi l x_j) \equiv 0.$$

This identity should be fulfilled for any x_j, l. It means that

$$2\cos(\pi l \triangle x) + \delta(\triangle x)^2 - 2 = 0.$$

From this equation we obtain an estimate for δ

$$\delta = \frac{2}{(\triangle x)^2}(1 - \cos \pi l \triangle x) = \frac{4}{(\triangle x)^2} \sin^2 \frac{\pi l \triangle x}{2} < \frac{4}{(\triangle x)^2}, \quad l = 1, .., M-1$$

After this we consider (3.1) for w^n

$$w^{n+1} - w^n + \delta \triangle \tau (\theta w^{n+1} + (1-\theta) w^n) = 0,$$

or, equivalently,

$$w^{n+1} = q_l w^n$$

with the amplification factor $q_l = \frac{1 - \delta \triangle \tau (1-\theta)}{(1 + \delta \triangle \tau \theta)}$.

The scheme is stable if $|q_l| \leq 1$, i.e.

$$-1 \leq \frac{1 - \delta \triangle \tau (1-\theta)}{(1 + \delta \triangle \tau \theta)} \leq 1. \tag{3.3}$$

Assume that $\delta > 0$ then the left side of the inequality (3.3) can be written in the following form

$$-1 - \delta \triangle \tau \theta \leq 1 - \delta \triangle \tau (1 - \theta),$$

$$\delta \triangle \tau (1 - 2\theta) \leq 2; \quad \theta \geq \frac{1}{2} - \frac{1}{\delta \triangle \tau}.$$

From the estimation of $\delta \leq \frac{4}{(\triangle x)^2}$ we can conclude that the scheme is stable for all $\triangle \tau > 0$ (unconditionally stable), when

$$\theta \geq \theta_0 = \frac{1}{2} - \frac{(\triangle x)^2}{4 \triangle \tau} = \frac{1}{2} - \frac{1}{4\gamma}.$$

For $0 \leq \theta < \theta_0$ the scheme will be stable if

$$\triangle \tau \leq \frac{(\triangle x)^2}{2(1 - 2\theta)},$$

or equivalently $\gamma \leq \frac{1}{2(1-2\theta)}$. In this case the scheme is called conditionally stable.

For Method 1 and Method 2 θ is equal to $\theta = \frac{1}{2} - \frac{1}{12\gamma} > \theta_0$. It means that these methods for interior points are unconditionally stable.

Note that the Crank-Nicolson scheme is also unconditionally stable ($\theta = \frac{1}{2}$).

Another method to analyze the stability of the scheme is the formal Fourier analysis. This method is based on the Fourier expansion on a one time level. The computational algorithm will be stable if some components in the Fourier expansion will decrease when moving to the next time level. We analyze the Crandall-Douglas method by the Fourier method. As we found out, the R3-methods and the Tangman's method are the same for our special case, so, we can generalize the results for all schemes.

We decompose the solution in the Fourier models $u_j^n = q^n e^{ikj\triangle x}$, where we denote by i - the imaginary unit and k-wave number. For the stability conditions we require $|q| \leq 1$. Put the expression for the u_j^n to the scheme and divide both parts by $q^n e^{ikj\triangle x}$

$$q\left((1-6\gamma)(e^{ik\triangle x}+e^{-ik\triangle x})+(10+12\gamma)\right)$$

$$=(1+6\gamma)(e^{ik\triangle x}+e^{-ik\triangle x})+(10-12\gamma).$$

$$q=\frac{(1+6\gamma)(e^{ik\triangle x}+e^{-ik\triangle x})+(10-12\gamma)}{(1-6\gamma)(e^{ik\triangle x}+e^{-ik\triangle x})+(10+12\gamma)}.$$

By using the identity $e^{ik\triangle x}+e^{-ik\triangle x}=2\cos k\triangle x$ we obtain

$$q=\frac{(1+6\gamma)(2\cos k\triangle x)+(10-12\gamma)}{(1-6\gamma)(2\cos k\triangle x)+(10+12\gamma)}.$$

For the stability of the θ-method the following inequality should hold

$$|q|\leq 1 \Leftrightarrow \left|\frac{(1+6\gamma)(2\cos k\triangle x)+(10-12\gamma)}{(1-6\gamma)(2\cos k\triangle x)+(10+12\gamma)}\right|\leq 1,$$

i.e.

$$-(1-6\gamma)(2\cos k\triangle x)-(10+12\gamma)\leq(1+6\gamma)(2\cos k\triangle x)+(10-12\gamma)$$

$$\leq(1-6\gamma)(2\cos k\triangle x)+(10+12\gamma).$$

After some calculations we obtain the stability condition $\gamma \geq 0$. This means that the optimal weighted θ-scheme is unconditionally stable.

The non-Compact method

We use the Fourier ansatz $u_j^n = q^n e^{ikj\triangle x}$ for the non-compact two-level five-point scheme. If we apply this substitution to the equation (2.47) and divide by $q^n e^{ikj\triangle x}$, we obtain

$$q=1+\gamma(e^{-ik\triangle x}-2+e^{ik\triangle x})$$

$$+\frac{\gamma^2}{2}(e^{-2ik\triangle x}-4e^{-ik\triangle x}+6-4e^{ik\triangle x}+e^{2ik\triangle x}).$$

We use the De Moivre's formulas to get trigonometric functions instead of exponential functions

$$q = 1 + 2\gamma\left(\cos(k\triangle x) - 1\right) + \gamma^2\left(-4\cos(k\triangle x) + 3 + \cos(2k\triangle x)\right). \quad (3.4)$$

For the stability conditions we should choose γ such that the inequality

$$|q| \le 1 \quad \Leftrightarrow \quad -1 \le q \le 1$$

is fulfilled.

First, we consider the right hand side of this inequality. Using some trigonometric formulas we rewrite it in the following form

$$1 - 4\gamma\sin^2(\frac{k\triangle x}{2}) + \gamma^2(\cos(2k\triangle x) - \cos(k\triangle x) + 6\sin^2(\frac{k\triangle x}{2})),$$

$$4\sin^2(\frac{k\triangle x}{2}) \ge \gamma(\cos(2k\triangle x) - \cos(k\triangle x) + 6\sin^2(\frac{k\triangle x}{2})),$$

i.e.

$$\gamma \le \frac{4\sin^2(\frac{k\triangle x}{2})}{\cos(2k\triangle x) - \cos(k\triangle x) + 6\sin^2(\frac{k\triangle x}{2})}. \quad (3.5)$$

Now we consider the numerator of (3.5)

$$4\sin^2(\frac{k\triangle x}{2}) = -2\cos(k\triangle x) + 2 = 2(1 - \cos(k\triangle x)). \quad (3.6)$$

In the denominator we obtain after some transformations

$$\cos(2k\triangle x) - \cos(k\triangle x) + 6\sin^2\left(\frac{k\triangle x}{2}\right) =$$

$$2\cos^2(k\triangle x) - 4\cos(k\triangle x) + 2 = (\sqrt{2}\cos(k\triangle x) - \sqrt{2})^2 =$$
$$2(\cos(k\triangle x) - 1)^2. \quad (3.7)$$

We insert both expressions (3.6) and (3.7) into (3.5) and get

$$\gamma \le \frac{2(1 - \cos(k\triangle x))}{2(\cos(k\triangle x) - 1)^2},$$

or, equivalently,

$$\gamma \le \frac{1}{1 - \cos(k\triangle x)}. \quad (3.8)$$

This stability condition (3.8) should be fulfilled for all $\triangle x$. Hence, we should take the most strict condition. Because of that we require $\cos(k\triangle x) = -1$, then from (3.8) we conclude that

$$\gamma \leq \frac{1}{2}.$$

Hence, the non-compact two-level five-point scheme for the heat equation (2.47) is conditionally stable. We can't use $\gamma > \frac{1}{2}$, because in that case we obtain strong oscillations for some modes and the method will not converge to an exact solution.

3.2 The order of the approximation

The second important parameter of the valuation of the scheme is the order of approximation. The heat equation is transformed to the system of linear algebraic equations. This trasformation is the source of the errors. The value of this error shows the order of approximation, i.e. the order of approximation is how accurate we discretize the equation. To find the order of approximation we use the Taylor series expansion.

We consider the point $(x_j, \tau^n + \frac{\Delta\tau}{2})$ as a central point in our scheme. The error of approximation is denoted by $\psi_j^{n+\frac{1}{2}}$. It is calculated by

$$\psi_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} - u_j^n}{\Delta\tau} - \theta D_{xx} u_j^n - (1-\theta) D_{xx} u_j^{n+1}. \tag{3.9}$$

Then we apply a Taylor series expansion for this point to the θ-scheme (2.37). For the time derivative the central difference scheme is used, it has the second order of approximation

$$\frac{u_j^{n+1} - u_j^n}{\Delta\tau} = \frac{\partial u}{\partial\tau}(x_j, \tau^n + \frac{\Delta\tau}{2}) + \mathcal{O}(\Delta\tau^2). \tag{3.10}$$

For the approximation the space derivatives we use the central scheme of the second order. On the n-th and the $(n+1)$-th time level we obtain correspondingly

$$D_{xx} u_j^n = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = (u_{xx} + \frac{\Delta x}{12} u_{xxxx})|_{\tau=\tau^n} + \mathcal{O}(\Delta x^4), \tag{3.11}$$

$$D_{xx} u_j^{n+1} = \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2}$$

$$= (u_{xx} + \frac{\Delta x}{12} u_{xxxx})|_{\tau=\tau^{n+1}} + \mathcal{O}(\Delta x^4). \tag{3.12}$$

Now we insert the expressions (3.10)-(3.12) into (3.9)

$$\psi_j^{n+\frac{1}{2}} = u_\tau + \mathcal{O}(\Delta\tau^2) - u_{xx} + \left[(\frac{1}{2} - \theta)\Delta\tau - \frac{\Delta x^2}{12} \right] u_{xxxx} + \mathcal{O}(\Delta x^4).$$

Because the structure of the heat equation, $u_\tau - u_{xx} = 0$, the error of approximation is equal to

$$\psi_j^{n+\frac{1}{2}} = \left[(\frac{1}{2} - \theta)\Delta\tau - \frac{\Delta x^2}{12} \right] u_{xxxx} + \mathcal{O}(\Delta\tau^2) + \mathcal{O}(\Delta x^4). \tag{3.13}$$

If the expression in the squared brackets in (3.13) is equal to zero, the approximation error is of fourth order in space and the second order in time

$$\left(\frac{1}{2} - \theta \right) \Delta\tau - \frac{\Delta x^2}{12} = 0,$$

i.e.,

$$\theta = \frac{1}{2} - \frac{\Delta x^2}{12\Delta\tau} = \frac{1}{2} - \frac{1}{12\gamma},$$

which is the optimal weight θ.

3.3 The numerical Results

The considered schemes belong to the large class of so-called θ-methods. For example, the fully explicit and implicit methods, and certainly Crank-Nicolson schemes are included in this class. We propose several numerical examples to compare the calculation of the option value in American case by using the finite difference schemes with different boundary conditions. The comparison is made with respect to the error and the accuracy of the numerical solution.

For all examples we consider the American call option. After the standard transformation for the Black-Scholes equation we obtain the heat equation on the unbounded interval. But the realization of the computational algorithms can not be made on the unbounded domain. We find the solution of the heat equation on the closed interval $[x_m, x_f(\tau)]$, where $x_m = a$ (the number a is strictly negative real number). On each time step we calculate the free-boundary value and magnify the computational domain.

We compare numerical results with an exact option value, which is a-priory unknown. The binomial method is very accurate therefore, we can take it to obtain the exact solution. The binomial method uses a large number of the steps to achieve the high order of accuracy. In our numerical examples the number of the steps is equal to 5000. For the numerical tests we use MATLAB program.

Example 1.

We consider a six-month American call option. The dividend payment $D = 0.03$ and the risk-free rate $r = 0.03$. The exercise price of this option is equal to 100$. The volatility σ^2 is 0.4. At the initial time the right boundary of the closed interval is calculated from the equation (2.13). A very important coefficient for the numerical schemes is the parabolic mesh ratio that is set to $\gamma = 1$ for all compact methods. In the case of the non-compact method which is conditionally stable for $\gamma \leq \frac{1}{2}$ we can't take this value, so we use $\gamma = \frac{1}{4}$. The number of the time steps is $N = 400$. The size of the space steps is defined from the expression

$$\gamma = \frac{\triangle\tau}{(\triangle x)^2} \quad \Rightarrow \triangle x = \sqrt{\frac{\gamma}{\triangle\tau}}.$$

We consider the different left boundary in the closed interval $[x_m, x_f]$. In the first case $x_m = a = -1$. The asset price for this $a = -1$ is approximately equal to 36.7879. For $x_m = a = -0.6$ the asset price is equal to 54.88. And for $x_m = a = -1$ is equal to 81.87. The Tables below show the results of the finite difference methods with different boundary conditions. We analyze influence of the left boundary to the solution.

Note, that the two-level five-point scheme is conditionally stable. This scheme is stable for $\gamma \leq \frac{1}{2}$. In the first numerical test $\gamma = 1$. For that value Heun's method has an oscillation of the solution. Therefore the numerical results by using this scheme are not included in the Table 3.1. In the Table 3.1 there are presented the intermediate results for the compact schemes and for the Crank-Nicolson method for three different boundary conditions.

As we see from the Tables 3.1-3.6, the numerical test confirms the theoretical results. The Method 1 and the Method 2 are the same and give the same solution to the heat equation. Also, R3A and R3B give the same results. So, we can conclude that all two-level three-point compact schemes are coincide for this one type of partial differential equations, for the heat equation. If we consider the Non-linear Black-Scholes model, as in [2], then the different high order compact schemes are different because of the first derivatives in the transformed equation.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	True Value
36.7879	Mayfield	0.0024	0.0023	0.0023	0.0015
	Han-Wu	0.0023	0.0023	0.0023	
	DTBC	0.0023	0.0023	0.0023	
62.5002	Mayfield	0.6108	0.6096	0.6096	0.4388
	Han-Wu	0.6108	0.6096	0.6096	
	DTBC	0.6108	0.6096	0.6096	
87.8095	Mayfield	6.1291	6.1234	6.1234	5.4801
	Han-Wu	6.1291	6.1234	6.1234	
	DTBC 6.1291		6.1234	6.1234	
110.5171	Mayfield	18.3457	18.3349	18.3349	17.6184
	Han-Wu	18.3457	18.3349	18.3349	
	DTBC	18.3457	18.3349	18.3349	
140.4948	Mayfield	42.3819	42.3748	42.3748	41.9937
	Han-Wu	42.3819	42.3748	42.3748	
	DTBC	42.3819	42.3694	42.3694	

Figure 3.1: *Value of the option in Example 1. Test 1a. We take the free boundary value a=-1 and the parabolic mesh ratio* $\gamma = 1$.

Asset price	*BC*	*C-N*	*Method 1*	*Method 2*	Heun	True Value
36.7879	Mayfield	0.0015	0.0015	0.0015	0.0016	0.0015
	Han-Wu	0.0015	0.0014	0.0014	0.0015	
	DTBC	0.0015	0.0014	0.0014		
63.1284	Mayfield	0.5699	0.5631	0.5631	0.5656	0.4803
	Han-Wu	0.5699	0.5631	0.5631	0.5656	
	DTBC	0.5699	0.5631	0.5631		
86.9358	Mayfield	5.4814	5.4543	5.4543	5.4631	5.1639
	Han-Wu	5.4814	5.4543	5.4543	5.4631	
	DTBC	5.4814	5.4543	5.4543		
110.5171	Mayfield	17.9773	17.9320	17.9320	17.9498	17.6184
	Han-Wu	17.9773	17.9320	17.9320	17.9498	
	DTBC	17.9773	17.9320	17.9320		
140.4948	Mayfield	42.1728	42.1254	42.1254	42.1480	41.9937
	Han-Wu	42.1728	42.1254	42.1254	42.1480	
	DTBC	42.1728	42.1254	42.1254		

Figure 3.2: *Value of the option in Example 1. Test 1b. We take the free boundary value a=-1 and the parabolic mesh ratio* $\gamma = \frac{1}{4}$.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	True Value
54.8812	Mayfield	0.1988	0.1980	0.1980	0.1247
	Han-Wu	0.1947	0.1940	0.1940	
	DTBC	0.1941	0.1933	0.1933	
86.0708	Mayfield	5.4820	5.4667	5.4667	4.8598
	Han-Wu	5.4819	5.4766	5.4766	
	DTBC	5.4819	5.4766	5.4766	
110.5171	Mayfield	18.3457	18.3349	18.3349	17.6184
	Han-Wu	18.3457	18.3349	18.3349	
	DTBC	18.3457	18.3349	18.3349	
140.4948	Mayfield	42.3819	42.3694	42.3694	41.9937
	Han-Wu	42.3819	42.3694	42.3694	
	DTBC	42.3819	42.3694	42.3694	
180.3988	Mayfield	80.4207	80.4017	80.4017	80.4046
	Han-Wu	80.4207	80.4017	80.4017	
	DTBC	80.4207	80.4017	80.4017	

Figure 3.3: *Value of the option in Example 1. Test 2a. We take the free boundary value a=-0.6 and the parabolic mesh ratio* $\gamma = 1$.

Asset price	*BC*	*C-N*	*Method 1*	*Method 2*	*Heun*	True Value
54.8812	Mayfield	0.1664	0.1639	0.1639	0.1652	0.1247
	Han-Wu	0.1590	0.1563	0.1563	0.1576	
	DTBC	0.1580	0.1555	0.1555		
86.9358	Mayfield	5.4815	5.4544	5.4544	5.4632	5.1637
	Han-Wu	5.4814	5.4543	5.4543	5.4631	
	DTBC	5.4814	5.4543	5.4543		
110.5171	Mayfield	17.9773	17.9320	17.9320	17.9498	17.6184
	Han-Wu	17.9773	17.9320	17.9320	17.9498	
	DTBC	17.9773	17.9320	17.9320		
140.4948	Mayfield	42.1728	42.1254	42.1254	42.1480	41.9937
	Han-Wu	42.1728	42.1254	42.1254	42.1480	
	DTBC	42.1728	42.1254	42.1254		
178.6038	Mayfield	78.5995	78.5920	78.5920	78.6141	78.6174
	Han-Wu	78.5995	78.5920	78.5920	78.6141	
	DTBC	78.5995	78.5920	78.5920		

Figure 3.4: *Value of the option in Example 1. Test 2b. We take the free boundary value a=-0.6 and the parabolic mesh ratio* $\gamma = \frac{1}{4}$.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	True Value
81.8731	Mayfield	4.0615	4.0548	4.0548	3.5455
	Han-Wu	4.0151	4.0092	4.0092	
	DTBC	4.0041	3.9982	3.9982	
86.0708	Mayfield	5.4518	5.4462	5.4462	4.8598
	Han-Wu	5.4150	5.4098	5.4098	
	DTBC	5.4046	5.3997	5.3997	
100.0000	Mayfield	11.8441	11.8356	11.8356	11.1068
	Han-Wu	11.8270	11.8185	11.8185	
	DTBC	11.8207	11.8126	11.8126	
140.4948	Mayfield	42.3810	42.3684	42.3684	41.9937
	Han-Wu	42.3795	42.3670	42.3670	
	DTBC	42.3787	42.3663	42.3663	
178.6038	Mayfiel d	78.6417	78.6232	78.6232	78.6174
	Han-Wu	78.6417	78.6231	78.6231	
	DTBC	78.6416	78.6231	78.6231	

Figure 3.5: *Value of the option in Example 1.Test 3a. We take the free boundary value a=-0.2 and the parabolic mesh ratio* $\gamma = 1$.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	*Heun*	True Value
81.8731	Mayfield	3.8914	3.8712	3.8712	3.8797	3.5455
	Han-Wu	3.7860	3.7645	3.7645	3.7724	
	DTBC	3.7728	3.7534	3.7534		
86.9358	Mayfield	5.5385	5.5108	5.5108	5.5197	5.1639
	Han-Wu	5.4555	5.4283	5.4283	5.4368	
	DTBC	5.4457	5.4182	5.4182		
100.0000	Mayfield	11.5111	11.4727	11.4727	11.4862	11.1068
	Han-Wu	11.4703	11.4322	11.4322	11.4456	
	DTBC	11.4643	11.4262	11.4262		
140.4948	Mayfield	42.1753	42.1279	42.1279	42.1505	41.9937
	Han-Wu	42.1721	42.1248	42.1248	42.1473	
	DTBC	42.1714	42.1241	42.1241		
178.6038	Mayfield	78.5954	78.5938	78.5938	78.6143	78.6174
	Han-Wu	78.5995	78.5920	78.5920	78.6141	
	DTBC	78.5995	78.5919	78.5919		

Figure 3.6: *Value of the option in Example 1. Test 3b. We take the free boundary value a=-0.2 and the parabolic mesh ratio* $\gamma = \frac{1}{4}$.

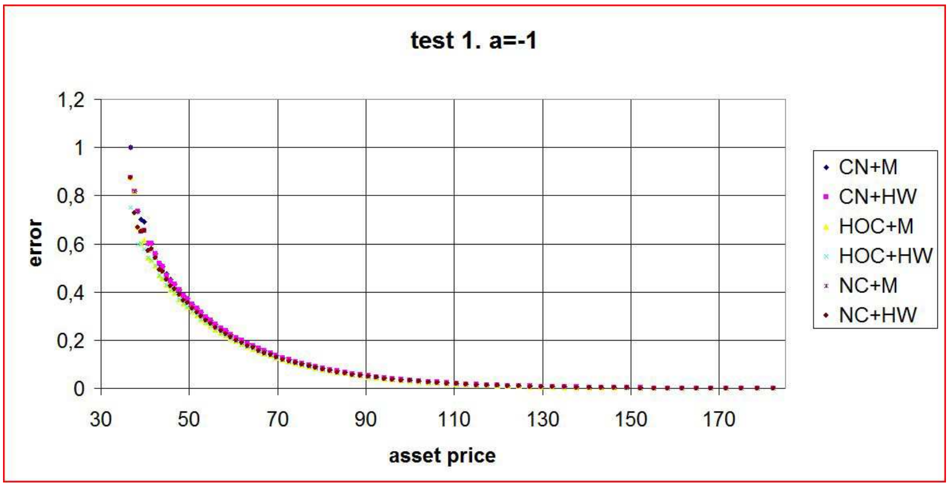

Figure 3.7: *The fractional error for example 1. We take the left boundary value a=-1 and the parabolic mesh ratio is equal to* $\gamma = \frac{1}{4}$.

The using different types of boundary conditions leads to changing of the stability properties. Furthermore, the accuracy of the solution depends on boundary conditions. A combination of the Crank-Nicolson stencil with Mayfield's BC gives comparatively an inaccurate solution in comparison to the same scheme but with the Han and Wu and the discrete TBC conditions.

The left boundary in the closed interval $[x_m, x_f]$ should be a negative number. Point out that if we put the left boundary considerably far from the zero point then the influence of different boundary conditions is not so visible. But if the left boundary is closer to zero then the influence is increasing and we see the difference between the Mayfield's, the Han and Wu and the discrete TBC conditions. For example, if $a = -1$, $\gamma = \frac{1}{4}$ (Fig. 3.2) for the asset price $S = 36.7879$, which is here a boundary value, the Crank-Nicolson method with the Mayfield's, the Han and Wu and the discrete TBC conditions give the same result. The difference between the application these BC for all other methods is just equal to 10^{-4}. But the increasing of value a involves growth of this difference. In the case $a = -0.2$ (Fig. 3.6) the margin between BC for all methods (including the Crank-Nicolson scheme) is approximately 0.1. The most accurate solution can be obtained by using the optimal weighted scheme with the discrete TBC boundary conditions. Note that the Heun's method is more accurate then the Crank-Nicolson scheme for all considered numerical tests.

Example 2.

In the second example we consider a 3-year American call option. The dividend payment $D = 0.07$. Other parameters of the problem are similar to the parameters values in the first example. The risk-free rate is equal to 0.03. The exercise price of this option is equal to 100$. The volatility σ^2 is 0.4. At the initial time the right boundary of the closed interval is calculated using the equation (2.13). The parabolic mesh ratio is set to $\gamma = 1$. In the case of the non-compact method which is conditionally stable for $\gamma \leq \frac{1}{2}$ we can not take this value, so we use $\gamma = \frac{1}{4}$. The number of time steps is $N = 400$.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	True Value
36.7879	Mayfield	0.9376	0.9365	0.9365	0.8121
	Han-Wu	0.9201	0.9190	0.9190	
	DTBC	0.9188	0.9178	0.9178	
63.0582	Mayfield	5.8760	5.8730	5.8730	5.5333
	Han-Wu	5.8708	5.8678	5.8678	
	DTBC	5.8703	5.8673	5.8673	
88.8533	Mayfield	15.6634	15.6589	15.6589	15.1843
	Han-Wu	15.6615	15.6570	15.6570	
	DTBC	15.6613	15.6568	15.6568	
110.7683	Mayfield	27.4561	27.4510	27.4510	26.9570
	Han-Wu	27.4553	27.4502	27.4502	
	DTBC	27.4552	27.4501	27.4501	
141.5127	Mayfield	48.5037	48.4988	48.4988	48.1016
	Han-Wu	48.5033	48.4985	48.4985	
	DTBC	48.5033	48.4985	48.4985	
199.4010	Mayfield	99.3558	99.3508	99.3508	99.4010
	Han-Wu	99.3558	99.3597	99.3597	
	DTBC	99.3558	99.3507	99.3597	

Figure 3.8: *Value of the option in Example 2. We take the free boundary value a=-1 and the parabolic mesh ratio* $\gamma = 1$.

By comparison both examples we conclude, that the boundary conditions are an important source of errors. As we see from the Tables 3.1 - 3.8, the error on the first several points is quite big and it goes down with the increasing of the asset price, i.e. with removal from the boundary.

The second example confirms the results about the quality of the boundary conditions. The high order compact scheme (HOC) with the discrete TBC boundary condition is the most acceptable for the American option pricing problem. Heun's non-compact method is better than the Crank-

Asset price	*BC*	*C-N*	*Method 1*	*Method 2*	*Heun*	True Value
36.7879	Mayfield	0.9041	0.9004	0.9004	0.9027	0.8121
	Han-Wu	0.8692	0.8656	0.8656	0.8678	
	DTBC	0.8674	0.8634	0.8634		
63.0582	Mayfield	5.7138	5.7033	5.7033	5.7088	5.5333
	Han-Wu	5.7038	5.6934	5.6934	5.6988	
	DTBC	5.7031	5.6929	5.6926		
88.8533	Mayfield	15.4173	15.4017	15.4017	15.4094	15.1843
	Han-Wu	15.4139	15.3983	15.3983	15.4059	
	DTBC	15.4136	15.3979	15.3979		
113.5151	Mayfield	28.8653	28.8474	28.8474	28.8563	28.6419
	Han-Wu	28.8639	28.8460	28.8460	28.8549	
	DTBC	28.8638	28.8458	28.8458		
145.0219	Mayfield	50.9588	50.9409	50.9409	50.9501	50.8152
	Han-Wu	50.9583	50.9404	50.9404	50.9496	
	DTBC	50.9583	50.9403	50.9403		
204.3457	Mayfield	104.1187	104.1079	104.1079	104.1148	104.3457
	Han-Wu	104.1186	104.1078	104.1078	104.1148	
	DTBC	104.1186	104.1078	104.1078		

Figure 3.9: *Value of the option in Example 2. We take the free boundary value a=-1 and the parabolic mesh ratio* $\gamma = \frac{1}{4}$.

Asset price	*BC*	*Crank-Nicolson*	*Method 1*	*Method 2*	True Value
81.8731	Mayfield	12.5815	12.5766	12.5766	12.0979
	Han-Wu	12.4424	12.4376	12.4376	
	DTBC	12.4255	12. 4219	12.4219	
109.8492	Mayfield	26.9158	26.9100	26.9100	26.4060
	Han-Wu	26.8167	26.8112	26.8112	
	DTBC	26.8006	26.7963	26.7963	
130.3956	Mayfield	40.3452	40.3393	40.3393	39.8942
	Han-Wu	40.2717	40.2661	40.2661	
	DTBC	40.2584	40.2540	40.2540	
170.7187	Mayfield	72.5266	72.5216	72.5216	72.3235
	Han-Wu	72.4958	72.4908	72.4908	
	DTBC	72.4892	72.4852	72.4852	
188.2925	Mayfield	88.6242	88.6211	88.6211	88.5570
	Han-Wu	88.6084	88.6052	88.6052	
	DTBC	88.6049	88.6023	88.6023	
202.6502	Mayfield	102.6107	102.6099	102.6099	102.6502
	Han-Wu	102.6066	102.6057	102.6057	
	DTBC	102.6057	102.6049	102.6049	

Figure 3.10: *Value of the option in Example 2. We take the free boundary value a=-0.2 and the parabolic mesh ratio* $\gamma = 1$.

Asset price	*BC*	*C-N*	*Method 1*	*Method 2*	*Heun*	True Value
81.8731	Mayfield	12.5602	12.5480	12.5480	12.5530	12.0979
	Han-Wu	12.2568	12.2449	12.2449	12.2498	
	DTBC	12.2397	12.2269	12.2269		
109.8492	Mayfield	26.7933	26.7774	26.7774	26.7844	26.4060
	Han-Wu	26.5716	26.5560	26.5560	26.5628	
	DTBC	26.5549	26.5384	26.5384		
127.2404	Mayfield	37.9867	37.9700	37.9700	37.9778	37.6754
	Han-Wu	37.8114	37.7949	37.7949	37.8026	
	DTBC	37.7970	37.7795	37.7795		
170.7187	Mayfield	72.4055	72.3941	72.3941	72.4001	72.3235
	Han-Wu	72.3295	72.3180	72.3180	72.3240	
	DTBC	72.3226	72.3104	72.3104		
188.2925	Mayfield	88.5544	88.5479	88.5479	88.5514	88.5570
	Han-Wu	88.5145	88.5079	88.5079	88.5114	
	DTBC	88.5108	88.5039	88.5039		
197.7466	Mayfield	97.7155	97.7121	97.7121	97.7139	97.7602
	Han-Wu	97.6949	97.6915	97.6915	97.6933	
	DTBC	97.6930	97.6994	97.6894		

Figure 3.11: *Value of the option in Example 2. We take the free boundary value a=-0.2 and the parabolic mesh ratio* $\gamma = \frac{1}{4}$.

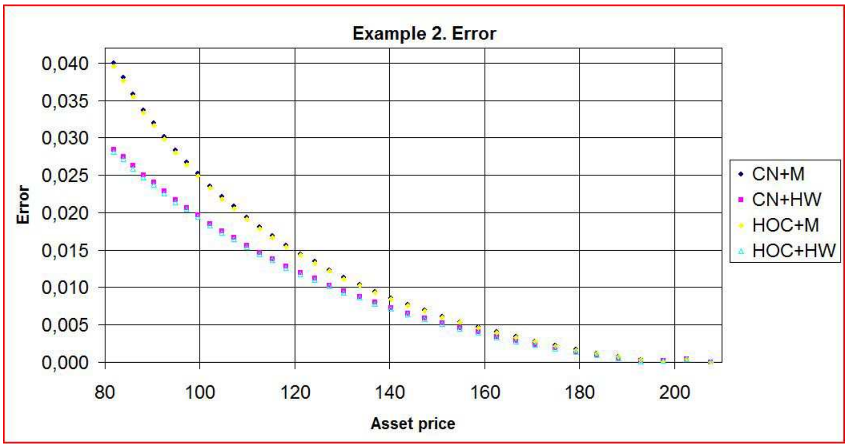

Figure 3.12: *The fractional error for example 1. We take the left boundary value a=-1 and the parabolic mesh ratio is equal to* $\gamma = \frac{1}{4}$.

Nicolson scheme. This non-compact method has one big disadvantage. It is stable only for certain values of the parabolic mesh ratio γ. Vice-versa, the Crank-Nicolson scheme is unconditionally stable.

Chapter 4

Conclusion

The main idea of our master thesis is the application different boundary conditions for compact and non-compact methods to the American option pricing problem and revelation the best combination between a scheme and the boundary conditions. We considered several high order compact schemes, which are different in the general case of diffusion-convection equation. The Black-Scholes equation for the American option is transformed to the heat equation. We found out that all considered high order compact schemes coincide for this equation. This theoretical result was confirmed by numerical tests.

The American option pricing problem is considered as a free boundary problem. We introduce an artificial boundary to limit the computational domain. In our thesis we applied the Dirichlet boundary conditions to the right boundary. In the Dirichlet boundary conditions we use exact solution. It means that the singular source of the error is the approximation of derivatives in the heat equation and there are no additional sources of errors.

For the interior points we use several difference methods. The high order compact and the Crank-Nicolson schemes are unconditionally stable. Heun's method is stable for the parabolic mesh ratio $\gamma \leq \frac{1}{2}$. It is one of the disadvantages of this method. We can not use $\gamma = 1$ for numerical tests, how it was proposed in many papers. The second disadvantage of the Heun's method is that we need 5 points on the n-th time level. Therefore, we have to introduce additional boundary conditions. For our numerical tests we take the Crank-Nicolson scheme as this additional equation. That is an improvement of this method. The numerical examples show, that this choice does not involve any computational costs or errors. The Heun's method with the Crank-Nicolson additional condition is more accurate than the Crank-Nicolson scheme itself.

After numerical tests we obtained that the best combination is the high order compact scheme with the Han and Wu boundary conditions on the left boundary. Using the different types of the boundary conditions do not lead to strong difference, but it is visible that the Han and Wu boundary conditions give smaller contribution to the error of approximation than Mayfield's boundary conditions. The influence of the boundary conditions decreases with removal from the left boundary, i.e. the accuracy of the numerical solution increase with the increasing of asset price.

All high order compact schemes coincide for the heat equation. They can be transformed to the optimal weighted scheme, or the optimal θ-method. This scheme has the fourth order in space and the second order in time. And there are no possibilities to create a new three-point two-level stencil for the heat equation, which will have a higher order of approximation. Therefore, the Method 1, the Method 2 and the R3-schemes are coincide in this case. New schemes can be obtained by using more points in stencil. It involves the loss of the density, i.e. the scheme will be not a compact scheme. Also, we can use a non-uniform grid. In that case the coefficients in the scheme depend on the space index j and the scheme has a new form. Another way to get the new method is creating compact schemes of the deferred order (for example, third order in space) with better properties like stability or monotonicity.

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